Constraint Solving: Methods and Applications

Dr. Serdar Kadioglu
serdark@cs.brown.edu

Brown University
Dept. of Computer Science
Combinatorial Problems

Explore a vast solution space towards a desired solution while trying to eliminate sub-parts of the solution space which are guaranteed not to have a (better) satisfying solution.
Combinatorial Problems

Explore a vast solution space towards a desired solution while trying to eliminate sub-parts of the solution space which are guaranteed not to have a (better) satisfying solution.

Interplay of search and inference
Branch & Bound

- A fundamental strategy
- Partition the problem (branching)
- Compute an estimate best solution for the sub-problem (bounding)
- Compare the bounds (pruning)
Branch & Bound

\[
\text{max } c^T x \\
s.t. \quad Ax \leq b \\
x_i \in \{0,1\}
\]
Branch & Bound

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\begin{align*}
\text{max } & \quad c^T x \\
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\end{align*}
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Classification of Techniques

- Branch & Bound
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- Branch & Bound
- Branch & Bound & Cut (Generates Cuts/Constraints)
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- Branch & Bound & Price (Generates Columns/Variables)
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- **Branch & Bound & Price & Cut**
Classification of Techniques

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- Branch & Bound & Price (Generates Columns/Variables)
- Branch & Bound & Price & Cut
- Branch & Bound & Price & Cut & Heuristics
**Agenda for Today**

1. Existing Paradigms / Solvers
2. Different Representations
3. Inference: Cutting Plane Examples
4. Branch & Bound Visualizations
5. Perspectives
Plethora of Paradigms

**Constraint Programming**
- Oracle ACT Solver
- IBM ILog CP
- Google OR-Tools
- Microsoft Solver Foundation

**Mathematical Programming**
- IBM ILog Cplex
- Dash Xpress-MP
- Gurobi Platform

**Boolean Satisfiability**
- Microsoft Z3
- MiniSAT
- SAT / SMT
- Predicate logic
  - First-order logic

**Rules**
- Production rule systems
- Oracle Business Rules, Jrules, JESS, Drools

**Heuristics**
- Low level local search
- Randomized search
- Meta-heuristics
  - Simulated annealing
  - Genetic algorithms
  - Ant-Colony systems
Simple Constraint Example

- You have 3 friends whose birthdays are approaching
- What presents to buy?
Simple Constraint Example

- You have 3 friends whose birthdays are approaching
- What presents to buy?
  1. Tix for SuperBowl
  2. Tix for Champions League final
  3. Tix for Boston Philharmonic
Simple Constraint Example

- You have 3 friends whose birthdays are approaching
- What presents to buy?
  1. Tix for SuperBowl
  2. Tix for Champions League final
  3. Tix for Boston Philharmonic
- **Different** present for each
Representation Integer Programming

- Decision Variables: $x_1, x_2, x_3$
- Domains: $x_1, x_2, x_3 \in \{1, 2, 3\}$
- Constraints: $x_1 \neq x_2$
Representation Integer Programming

- Decision Variables: $x_1, x_2, x_3$
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Can we actually do this?
Representation Binary Integer Programming

- Decision Variables: \{x_{11}, x_{12}, x_{13}\}, \{x_{21}, x_{22}, x_{23}\}, \{x_{31}, x_{32}, x_{33}\}
- Domains: \ x \in \{0,1\}
- Constraints:
Representation Binary Integer Programming

- Decision Variables: \{x_{11}, x_{12}, x_{13}\}, \{x_{21}, x_{22}, x_{23}\}, \{x_{31}, x_{32}, x_{33}\}
- Domains: \(x \in \{0, 1\}\)
- Constraints: \(x_{11} + x_{21} + x_{31} = 1\) for value 1
**Representation Binary Integer Programming**

- **Decision Variables:** \{x_{11}, x_{12}, x_{13}\}, \{x_{21}, x_{22}, x_{23}\}, \{x_{31}, x_{32}, x_{33}\}
- **Domains:** \(x \in \{0, 1\}\)
- **Constraints:**
  - \(x_{11} + x_{21} + x_{31} = 1\) for value 1
  - \(x_{11} + x_{12} + x_{13} = 1\) for variable 1
  - ...
**Representation**  Boolean Satisfiability

- **Decision Variables:** \( \{x_{11}, x_{12}, x_{13}\}, \{x_{21}, x_{22}, x_{23}\}, \{x_{31}, x_{32}, x_{33}\} \)
- **Domains:**  \( x \in \{T/F\} \)
- **Constraints:**
**Representation Boolean Satisfiability**

- Decision Variables: \{x_{11}, x_{12}, x_{13}\}, \{x_{21}, x_{22}, x_{23}\}, \{x_{31}, x_{32}, x_{33}\}
- Domains: \(x \in \{T/F\}\)
- Constraints: \(\bigwedge_{i,j=1 \land i<j}^{n} x_i \neq x_j\) \(\forall i,j\) \text{ pair}
  \(x_i \neq x_j \equiv \bigvee_{i=1}^{l} (\neg x_{il} \lor x_{jl}) \land (x_{il} \lor \neg x_{jl})\) \(\forall \text{ value } l\)
Representation Constraint Programming

- Decision Variables: $x_1, x_2, x_3$
- Domains: $\{1, 2, 3\}$
- Constraints: $x_i \neq x_j \quad \forall \ i,j$ pair
Representation Constraint Programming

- Decision Variables: $x_1, x_2, x_3$
- Domains: $\{1, 2, 3\}$
- Constraints: $\text{AllDifferent}(x_1, x_2, x_3)$
Part – II

Cut Generation

Source:
Some examples are from Ed Rothberg, The CPLEX Library: Presolve and Cutting Planes
Part – II
Cut Generation
Also related to presolve
Rounding Cuts

- Fractional bound on integer variables can be truncated
- Example \( x_1 + x_2 \leq 2.5 \)
- Inference \( x_1 + x_2 \leq 2 \)
- Effects can become non-trivial when combined
Surrogate Constraints

- Example
  \[ x_1 + 2x_2 \leq 3 \]
  \[ 2x_1 + x_2 \leq 4 \]

- Inference
  \[ 3x_1 + 3x_2 \leq 7 \]
  \[ x_1 + x_2 \leq 2 \]
GCD Reduction

- Given a constraint with all integer variables and integer coefficients
- Divide all by GCD
- LHS is integral, round off RHS

Example: $3x + 6y + 9z \leq 11$

Inference: $x + 2y + 3z \leq 3$
Knapsack Cover Cut

- Example: $3x_1 + 5x_2 + 4x_3 + 2x_4 + 7x_5 \leq 8$ \quad x \text{ binary integer}
- $3 + 5 + 2 > 8$ that is, $x_1$ and $x_2$ and $x_4$ cannot simultaneously be 1
- Inference: $x_1 + x_2 + x_4 \leq 2$
- Any other?
- Which one to use?
Lifting Constraints

- Given a constraint $\Sigma ax \geq b$ involving some binary $x_k$
- The question is; whether fixing $x_k = 1$ cause constraint to go slack?
- Then we can update the coefficient of $x_k$
- Example $2x + y \geq 1$
- Inference $x + y \geq 1$
Clique Cuts

- Two binary variables are incompatible if they cannot both be 1
- $x + y \leq 1$ means $x$ and $y$ are incompatible
- A clique is a set of pairwise incompatible variables
- Example: $x + y \leq 1$ and $x + z \leq 1$ and $y + z \leq 1$
- Inference: $x + y + z \leq 1$
Presolve Tricks

- Fixed variables
- Inactive constraints
  Example: \( x + y \leq 2 \) and \( x, y \) binary
- Redundant constraint
  Example: \( x + y \leq 2 \) and \( x + y \leq 3 \)
Presolve Tricks

- E.g. variables with
  - positive objective coefficients
  - belongs to only less-than-constraints
  - have all non-negative matrix coefficients
- These can be fixed to their lower bound
Presolve Impact

- Vital part of solving MIP models
- Significant improvement
  - Reduction in problem size
  - Reduction in runtime
Presolve Cplex Example

Initial LP formulation
\[\begin{align*}
e_1 & : \quad z = 2x_1 + x_2 - x_3; \\
e_2 & : \quad 5x_1 - 2x_2 + 8x_3 = 15; \\
e_3 & : \quad 8x_1 + 3x_2 - x_3 = g = 9; \\
e_4 & : \quad x_1 + x_2 + x_3 = l = 6; \\
x_1 & : \quad \text{up} = 3; \\
x_2 & : \quad \text{up} = 1; \\
x_3 & : \quad \text{lo} = 1;
\end{align*}\]

--- Generating LP model P1
--- wolsey_2.gms(25) 3 Mb
--- 4 rows 4 columns 13 non-zeros
--- Executing CPLEX: elapsed 0:00:00.017

Cplex 12.2.0.0, GAMS Link 34

Reading data...
Starting Cplex...
Tried aggregator 1 time.
LP Presolve eliminated 4 rows and 4 columns.
All rows and columns eliminated.
Presolve time = 0.00 sec.
LP status(1): optimal

Optimal solution found.
Objective: 3.600000
Applying Cuts

- Cuts increase the size of the model!
- Solving the linear relaxation after each cut is expensive
- Try different rounds of cutting plane generation
- Limit the number of cuts per round
Applying Cuts

- Cuts increase the size of the model!
- Solving the linear relaxation after each cut is expensive
- Try different rounds of cutting plane generation
- Limit the number of cuts per round

Balance Inference and Search
Most Successful Cuts for Generic Problems

[Bixby, Rothberg (2007)]

<table>
<thead>
<tr>
<th>Disabled Cut</th>
<th>Year</th>
<th>Mean Performance Degradation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gomory Mixed Integer (GMIC)</td>
<td>1960</td>
<td>2.52X</td>
</tr>
<tr>
<td>Mixed Integer Rounding</td>
<td>2001</td>
<td>1.83X</td>
</tr>
<tr>
<td>Knapsack Cover</td>
<td>1983</td>
<td>1.40X</td>
</tr>
<tr>
<td>Flow Cover</td>
<td>1985</td>
<td>1.22X</td>
</tr>
<tr>
<td>Implied Bound</td>
<td>1991</td>
<td>1.19X</td>
</tr>
<tr>
<td>Flow path</td>
<td>1985</td>
<td>1.04X</td>
</tr>
<tr>
<td>Clique</td>
<td>1983</td>
<td>1.02X</td>
</tr>
<tr>
<td>GUB Cover</td>
<td>1998</td>
<td>1.02X</td>
</tr>
<tr>
<td>Disjunctive</td>
<td>1979</td>
<td>0.53X</td>
</tr>
</tbody>
</table>
Branch & Bound Visualization

Source:
Brady Hunsaker, Visualization of B&B Algorithms
Branch & Bound Visualization

B&B tree (dataset2per8inv10.dat 7s )

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Brady Hunsaker, Visualization of B&B Algorithms
Branch & Bound Visualization

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Part – III
Perspectives
Analytics Perspective

Descriptive analytics: What happened?
Predictive analytics: What will happen?
Prescriptive analytics: What should I do about it?

Source:
Analytics Perspective

Data
- Hybrid
- Structured
- Unstructured

Analytics
- Descriptive
- Diagnostic
- Predictive
- Prescriptive

Decision
- Online/Offline
- Real-time
- Embedded
Collaborative Approach
Combination of Techniques

- Logic-based Benders’ Decomposition: MP + CP
- Warm started: CP + MP
- Local Search and MetaHeuristics + CP
- Column Generation, Branch & Price
- Instance-Specific Learning Algorithms
- Algorithm Configuration / Parallelization
Dynamic Adaptive Branching

Source:
G. Di Liberto, S. Kadioglu, K. Leo, Y. Malitsky, DASH: Dynamic Approach for Switching Heuristics
**Concluding Remarks**

- Optimization Technology is key
- Changing requirements dynamic environments
- High-level solutions are vital for long-term success
- User interaction is important