CSCI 1951-G – Optimization Methods in Finance
Part 02: Asset Pricing

February 12, 2016
Roadmap

1. Introduce derivatives and options
2. Define arbitrage and rational pricing
3. Formulate the problem of asset pricing
4. Use replication to solve asset pricing
5. Introduce risk-neutral probabilities to solve asset pricing
6. Prove the first fundamental theorem of asset pricing
7. Discuss how to detect arbitrage opportunities via LP
Derivatives and options

A security is any *tradeable financial asset* (e.g., a stock, a bond, ...)

**Definition (Derivative)**

A security whose price depends on the value of another *underlying security*.

**Definition ((European) (call) option)**

An *European call option* gives the holder the right to purchase an underlying security at a prescribed amount (the *strike price*) at a prescribed time in the future (the *expiration date*).

- An *American* call option gives the holder to purchase the underlying security even before the expiration date.
- An American/European *put* options gives the holder the right to *sell* the underlying security before/at the expiration date.
Example (Using options for hedging)

- I own some ACME shares, that currently trade for $40.
- I predict that the ACME price on 3/5 will be either $20 with probability $p$ or $80$ with probability $(1 - p)$.
- I buy an European call option for one ACME share at $50$ for 3/5.
- I now have the right to buy one share of ACME at $50$ on 3/5, no matter the real trading price of ACME on 3/5.
- I don’t have to exercise my right: if ACME price is $20$ on 3/5, I won’t buy!

Question (Pricing problem)

*How much should I pay for the option? What is a fair price for it?*

The price should depend on the current trading price of ACME and on its behavior in the future.
What is a “fair price”? 

Assumptions:

- there is a risky asset (the ACME stock) and a riskless asset (cash);
- the interest rate for borrowing/lending cash is the same and constant. We call it the risk-free interest rate;
- we can buy and sell fractional amounts of shares. We can sell short, i.e., sell shares that we do not even own;
- the market is frictionless: buying/selling shares and borrowing/lending cash has no transaction costs;
Temporary assumption: Arbitrage-free market

(Temporary) assumption: there is no arbitrage opportunity.

Definition (Arbitrage)

Arbitrage is an investment strategy that is profitable because of a price difference between two or more markets.

Example (Arbitrage)

• Assume the $/€ rate is 1.08 in Frankfurt, and 1.06 in NYC.
• You convert €1,000 to $1,080 in Frankfurt, and immediately convert $1,080 to €1,1018 in NYC.
• You made a profit of €18 by doing effectively nothing.
• The “strategy” of converting €→$→€ is arbitrage.

Arbitrage is like free food:
• you have the opportunity of getting something for nothing;
• if you take the opportunity, you can gain a lot (... of weight?);
• the opportunity goes away very quickly (the markets rebalance);
• Under the assumption of an arbitrage-free market,
  1. the law of one price holds: an asset has the same price on all markets; and
  2. assets with identical cash flows must trade at the same price

• The second item also means that:

**Important consequence**

Assets that have the same cash flows in all possible future scenarios, must trade at the same price now, otherwise there is an opportunity for arbitrage.

• Let’s exploit this fact (under the arbitrage-free market assumption) to price the option!
Assumption (from arbitrage-free market)

Assets that have the same cash flows in all possible future scenarios, must trade at the same price now.

Definition (Replication)

• Create a portfolio of securities with \textit{known current prices} (e.g., cash and ACME shares) that \textit{in each possible scenario} is worth as much as the payoff of the option in that scenario.

• From the assumption, we have that the option must cost \textit{now} as much as the portfolio costs \textit{now}.
Example: asset pricing through replication

Example (Cont.)

- We want to build a portfolio of $\Delta$ ACME shares and $B$ cash (two securities with known current prices) that, on 3/5, is worth as much as the payoff of the option:

<table>
<thead>
<tr>
<th>3/5 ACME Price ($)</th>
<th>80</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Option payoff ($)</td>
<td>80-50=30</td>
<td>0</td>
</tr>
</tbody>
</table>

- The portfolio is worth $40\Delta + B$ now. How about on 3/5?

<table>
<thead>
<tr>
<th>3/5 ACME Price ($)</th>
<th>80</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portfolio worth ($)</td>
<td>$80\Delta + B$</td>
<td>$20\Delta + B$</td>
</tr>
</tbody>
</table>

- We must solve (for $\Delta$ and $B$) the system:

$$
\begin{cases}
80\Delta + B = 30 \\
20\Delta + B = 0
\end{cases}
$$

The solution is $\Delta = 1/2$ and $B = -10$.

- Hence today the portfolio is worth $10$.

- Thus, the fair price of the option today is $10$. 
We now slightly generalize the example.

- \( S_0 \) is the current price of the underlying security.
- The price of the underlying security at a future time \( t \) is either
  \[ S^u_t = S_0 \cdot u \] or
  \[ S^d_t = S_0 \cdot d. \]
  Assume \( u > d \).
- There is fixed interest rate \( r \) on cash for the period \([0, t]\).
  Let \( R = 1 + r \) and assume \( d < R < u \) (otherwise there's arbitrage).
- The derivative security that has payoffs \( C^u_t \) and \( C^d_t \) respectively.
- We want to create a portfolio of \( \Delta \) shares of the underlying security and $B cash to replicate the payoffs of the derivative, in order to price it.
- What are the values of \( \Delta \) and \( B \)? We must solve the system

\[
\begin{align*}
\Delta S_0 \cdot u + BR &= C^u_t \\
\Delta S_0 \cdot d + BR &= C^d_t
\end{align*}
\]
Asset pricing through replication (cont.)

We must solve the following system for $\Delta$ and $B$:

$$
\begin{align*}
\Delta S_0 \cdot u + BR &= C_t^u \\
\Delta S_0 \cdot d + BR &= C_t^d
\end{align*}
$$

The solution of the system is

$$
\Delta = \frac{C_t^u - C_t^d}{S_0(u - d)}, \quad B = \frac{uC_t^d - dC_t^u}{R(u - d)}.
$$

The portfolio is worth $S_0\Delta + B$ today, so the price of the derivative must be the same:

$$
C_0 = \frac{C_t^u - C_t^d}{(u - d)} + \frac{uC_t^d - dC_t^u}{R(u - d)}.
$$

Observation

This price is independent from

- the probabilities that the price of the underlying security goes up or down;
- the current price $S_0$ (indeed the underlying stock may not even exist today!)
Risk-neutral probabilities

Let

\[ p_u = \frac{R - d}{u - d}, \quad p_d = \frac{u - R}{u - d} \]

Fact

We have \(0 < p_u, p_d < 1\) and \(p_u + p_d = 1\).
These quantities can be seen as probabilities, and are known as risk-neutral probabilities (RNPs).

We have \(C_0 = R^{-1}(p_u C_u^t + p_d C_d^t)\).
The division by \(R\) converts future values into current values.

Fact

The price of an option is today’s value of the expected payoff w.r.t. the RNPs, discounted by the risk-free interest rate.

Warning

The RNPs are only a mathematical device: they are completely unrelated to the probabilities of the different scenarios.
Risk and asset pricing

Fact

The price of an option is today’s value of the expected payoff w.r.t. the RNPs, discounted by the risk-free interest rate.

Question

Why isn’t the price of an option just the discounted expected payoff w.r.t. the “real” probabilities of the underlying price going up or down?

Answer

Because of risk: a risky investments is not as attractive as a riskless investment with the same price and same expected payoff.

• For a riskless asset, (e.g., cash), investors are willing to pay the discounted expected payoff w.r.t. the real probabilities.
• For a risky asset, investors want to pay less than the discounted expected payoff.
Risk and asset pricing (cont.)

Fact

Investors want to pay a risky asset less than the discounted expected payoff w.r.t. the real probabilities.

Consider a specific investor. She believes that the risky asset price will go up to $C_t^u$ with probability $p$ (down to $C_t^d$ with prob. $1 - p$).

Question

What price $S_0$ should she pay for the asset now?
Risk and asset pricing (cont.)

- Each investor has her own $p$, while $C^u_t$ and $C^d_t$ are fixed for everyone.

- Let $\eta(p)$ be the expected payoff w.r.t. $p$ and $1 - p$:
  \[
  \eta(p) = pC^u_t + (1 - p)C^d_t
  \]

- Let $\rho(p)$ be a risk measure of the risky asset, e.g., the variance of the payoff (the actual risk measure does not matter).

- The investor is susceptible to risk: she wants to pay a lower price that depends on the risk and on a (personal) risk aversion parameter $\lambda$. Instead of paying $\eta(p)/R$, she wants to pay
  \[
  S_0(p, \lambda) = \frac{\eta(p) - \lambda\rho(p)}{R}
  \]
Risk and asset pricing (cont.)

Each investor has her own \((p, \lambda)\) and computes her price \(S_0\) as

\[
S_0(p, \lambda) = \frac{\eta(p) - \lambda \rho(p)}{R}
\]

- We do not know how investors choose their \(p\) and \(\lambda\).
- But we made the arbitrage-free market assumption, so the law of one price holds: all investors must agree on a single price!
- Then \(S_0(p, \lambda)\) is a constant \(S_0\)

**Question**

Is there a \(p^*\) such that the price \(S_0\) is the discounted expected payoff w.r.t. \(p^*\) and \(1 - p^*\) (i.e., \(S_0 = \eta(p^*)/R\))?

If there is such a \(p^*\), then we can ignore the risk when computing the price of the risky asset, and act as if the investors were risk-neutral (i.e., do not care about risk).
The investors compute their prices as

\[ S_0 = S_0(p, \lambda) = \frac{\eta(p) - \lambda \rho(p)}{R} \]

**Question**

*Is there a \( p^\ast \) such that the price \( S_0 \) is the discounted expected payoff w.r.t. \( p^\ast \) and \( 1 - p^\ast \) (i.e., \( S_0 = \eta(p^\ast)/R \))?*

Since \( S_0 \) is a constant, so we can write \( \lambda \) as a function of \( p \):

\[ \lambda(p) = \frac{\eta(p) - S_0 R}{\rho(p)} \]

**Question (Reformulated)**

*Is there a \( p^\ast \) such that \( \lambda(p^\ast) = 0 \)?*

If there is such a \( p^\ast \), we have \( S_0 = \eta(p^\ast)/R \).
\[ \lambda(p) = \frac{\eta(p) - S_0 R}{\rho(p)} \]

**Question (Reformulated)**

*Is there a \( p^* \) such that \( \lambda(p^*) = 0 \)?*

**Question (Reformulated again)**

*Is there a \( p^* \) such that \( \eta(p^*) - S_0 R = 0 \)?*

It doesn’t matter what risk measure \( \rho(p) \) we are using!

**Answer**

\[ \eta(p^*) - S_0 R = 0 \iff pC^u_t + (1 - p)C^d_t - S_0 R = 0 \iff pS_0 u + (1 - p)S_0 d - S_0 R = 0 \]

The solution is

\[ p = \frac{R - d}{u - d} = p_u \quad (\text{hence } 1 - p = \frac{u - R}{u - d} = p_d) \]
The nature of the RNPs

Observation
The RNPs are values such that \textit{iff the real probabilities of up/down for the underlying security price were the RNPs}, then we could consider the investors as \textit{risk neutral}.

Observation
The price of the derivative is always the expected payoff \textit{w.r.t. the RNPs}, even if the real probabilities are not the RNPs, because the price set by the market is actually \textit{independent from the real probabilities}.
General setting

We now generalize a bit:

- Let $\Omega = \{\omega_1, \ldots, \omega_m\}$ be a set of possible disjoint events, (e.g., possible prices of a security at a future date $t$).
- Let $S^i$, $i = 1, \ldots, n$ be $n$ securities, and $S^0$ be a risk-free security (e.g., cash) that pays interest rate $r$ between now and date $t$.
- Let $S^i_0$ be the current known price of security $S^i$.
  We assume $S^0_0 = 1$.
- Let $S^i_t(\omega_j)$ be the price of security $i$ at time $t$ if event $\omega_j$ is verified. We assume $S^0_t(\omega_j) = R = 1 + r$, for all $j$. 
## Definition (Risk-neutral probability measure)

A risk-neutral probability measure on the set $\Omega = \{\omega_1, \ldots, \omega_m\}$ is a vector of positive numbers $(p_1, \ldots, p_m)$ such that

$$\sum_{j=1}^{m} p_j = 1$$

and for every security $S^i_t$, $i = 0, \ldots, n$, we have

$$S^i_0 = \frac{1}{R} \sum_{j=1}^{m} p_j S^i_t(\omega_j) = \frac{1}{R} \hat{E}[S^i_t]$$

were $\hat{E}[S^i_t]$ denotes the expected value of the r.v. $S^i_t$ w.r.t. the probability distribution $p_1, \ldots, p_m$. 

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**Note:**

- $\hat{E}$ represents the expected value.
- $p_j$ denotes the probability of the $j$th outcome.
- $S^i_0$ and $S^i_t$ are security values at time 0 and $t$, respectively.
The first fundamental theorem of asset pricing

A risk-neutral probability measure exists if and only if there is no arbitrage.
**Types of arbitrage**

**Type A:** an investment strategy that has positive cash flow initially, and no risk of a loss later.

**Example (Not a great one...)**

Someone on the street gives you money and never asks it back.

**Type B:** an investment strategy that has zero cost initially, no risk of a loss later, and a non-zero probability of a strictly positive payoff later.

**Example**

You “buy” a share of ACME for $0, and ACME may pay dividends in the future.
Proof preliminaries

Consider the following LP

\[
\min_x \sum_{i=0}^{n} S_0^i x_i \\
\sum_{i=0}^{n} S_t^i(\omega_j) x_i \geq 0, \quad j = 1, \ldots, m
\]

Fact (1)

Any feasible solution with negative objective value is a type-A arbitrage. If there is such a solution, the problem is unbounded.

Fact (2)

\(x = 0\) is a feasible solution with objective value 0. Hence the optimal objective value is non-positive.

Fact (3)

There is no type-A arbitrage if and only if the optimal objective value of the LP is 0.
Proof preliminaries (cont.)

Keep considering the LP

\[ \min_x \sum_{i=0}^{n} S_i^0 x_i \]

\[ \sum_{i=0}^{n} S_t^i(\omega_j) x_i \geq 0, \quad j = 1, \ldots, m \]

**Fact (4)**

*Any feasible (optimal) solution with objective value 0 such that there is an index \( j \) for which \( \sum_{i=0}^{n} S_t^i(\omega_j) x_i > 0 \), is a type-B arbitrage.*

**Fact**

*With small modifications to the LP we can detect arbitrage opportunities if they exist.*
Proof preliminaries (cont.)

Consider the dual of the previous LP:

\[
\begin{align*}
\max_p & \sum_{j=1}^{m} 0p_j \\
\sum_{j=1}^{m} S_i^j(\omega_j)p_j &= S_0^i, \quad i = 0, \ldots, n \\
p_j &\geq 0, j = 1, \ldots, m
\end{align*}
\]

**Fact (5)**

*The objective function is constant: any dual feasible solution is dual optimal.*

**Fact (6)**

*The constraints of the dual are essentially requiring \( q = (Rp_1, \ldots, Rp_j) \) to be a RNP measure, except that we are allowing \( p_j \geq 0 \), instead of \( p_j > 0 \).*
We will need the following result.

**Theorem (Strict complementarity – Goldman and Tucker, 1956)**

*When both the primal and dual LP have feasible solutions, there must be \( x^* \) and \( y^* \) optimal for the respective problems that are strictly complementary.*

In particular, if we let \( s = Ax^* - b \), then \( s_i = 0 \Leftrightarrow y_i^* > 0 \).
Actual proof of the theorem of asset pricing

Proof for “No arbitrage implies existence of RNP measure”:

- If there is no type-A arbitrage, then, from Facts 1 the primal is not unbounded, and, from Fact 2, it has a feasible solution and hence an optimal solution.

- From this and strong duality, then the dual must have a feasible solution \( p^* \). From Fact 5, \( p^* \) must be optimal.

- There is no type-B arbitrage, then, from Fact 4, all constraints of the primal are tight for an optimal solution \( x^* \) and \( s = Ax^* - b = 0 \).

- From this and strict complementarity, we then have that the optimal dual solution must be \( p^* > 0 \).

- From the dual constraint for \( i = 0 \) we get \( \sum_{j=1}^{m} p_j^* = 1/R \). By multiplying \( p_j^* \) by \( R \), we obtain a RNP measure. The proof for the other direction follows in a similar way.
Detecting arbitrage opportunities

Observation

Fast detection of arbitrage opportunities can lead to large gains.

• The primal LP corresponds to maximizing profit subject to a guarantee of no loss in any scenario.
• If this LP is unbounded, then there is type A arbitrage.
• If a LP is feasible, then it is unbounded iff the dual is infeasible.
• The RNPs are a feasible solution to the dual that shows that the primal is bounded.

By efficiently verifying if the dual is feasible, we can find whether there is arbitrage opportunity.

If we do it efficiently, we can make profits.