Expectation is not everything... 

With probability 0.99, you owe me $10; with probability 0.01, I owe you $1000. Would you play this game?
Expectation is not everything…

With probability $0.99$, you owe me $10$; with probability $0.01$, I owe you $1000$. Would you play this game?

$X = \text{your winning (in $)}. \ E[X] = 0.1.$

$Y = \text{my winning (in $)}. \ E[Y] = -0.1.$
Expectation is not everything...

With probability 0.99, you owe me $10; with probability 0.01, I owe you $1000. Would you play this game?

\[ X = \text{your winning (in $)}. \quad \mathbb{E}[X] = 0.1. \quad \text{But } \Pr(X \geq \mathbb{E}[X]) = 0.01 \]
(in fact, \( \Pr(X \geq 0) = 0.01 \))

\[ Y = \text{my winning (in $)}. \quad \mathbb{E}[Y] = -0.1. \quad \text{But } \Pr(Y \geq \mathbb{E}[Y]) = 0.99 \]
(in fact, \( \Pr(Y > 0) = 0.99 \))
Which algorithm would you prefer?

1. The expected run time of the algorithm is 1 hour but it will take 100 hours in 1% of the runs.
2. The run time is always 2 hours.

We need to bound the probability that the run time of the algorithm deviates significantly from its average.
Bounding Deviation from Expectation

**Theorem**

**Markov Inequality** For any non-negative random variable $X$, and for all $a > 0$,

$$Pr(X \geq a) \leq \frac{E[X]}{a}.$$ 

**Proof.**

$$E[X] = \sum_i iPr(X = i) \geq a \sum_{i \geq a} Pr(X = i) = aPr(X \geq a).$$

Example: What is the probability of getting more than $\frac{3N}{4}$ heads in $N$ coin flips? $\leq \frac{N/2}{3N/4} \leq \frac{2}{3}$. 
**Variance**

**Definition**

The **variance** of a random variable $X$ is


**Definition**

The **standard deviation** of a random variable $X$ is

$$\sigma(X) = \sqrt{Var[X]}.$$
Example: Let $X$ be a 0-1 random variable with $\Pr(X = 0) = \Pr(X = 1) = 1/2$.

$$E[X] = 1/2.$$ 

$$\text{Var}[X] = \frac{1}{2} \left( 1 - \frac{1}{2} \right)^2 + \frac{1}{2} \left( 0 - \frac{1}{2} \right)^2 = \frac{1}{4}.$$
Chebyshev’s Inequality

**Theorem**

*For any random variable* $X$, *and any* $a > 0$,

$$Pr(|X - E[X]| \geq a) \leq \frac{Var[X]}{a^2}.$$ 

**Proof.**

$$Pr(|X - E[X]| \geq a) = Pr((X - E[X])^2 \geq a^2)$$

By Markov inequality

$$Pr((X - E[X])^2 \geq a^2) \leq \frac{E[(X - E[X])^2]}{a^2}$$

$$= \frac{Var[X]}{a^2}$$
For any random variable $X$ and any $a > 0$:

$$Pr(|X - E[X]| \geq a\sigma[X]) \leq \frac{1}{a^2}.$$

For any random variable $X$ and any $\varepsilon > 0$:

$$Pr(|X - E[X]| \geq \varepsilon E[X]) \leq \frac{\text{Var}[X]}{\varepsilon^2 (E[X])^2}.$$
Theorem

If $X$ and $Y$ are independent random variables

$$E[XY] = E[X] \cdot E[Y].$$

Proof.

$$E[XY] = \sum_i \sum_j i \cdot j \Pr((X = i) \cap (Y = j)) = \sum_i \sum_j ij \Pr(X = i) \cdot \Pr(Y = j) = \left(\sum_i i \Pr(X = i)\right) \left(\sum_j j \Pr(Y = j)\right).$$
Theorem

If \( X \) and \( Y \) are independent random variables

\[
\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y].
\]

Proof.

\[
\text{Var}[X + Y] = E[(X + Y - E[X] - E[Y])^2] = \\
E[(X - E[X])^2 + (Y - E[Y])^2 + 2(X - E[X])(Y - E[Y])] = \\
\text{Var}[X] + \text{Var}[Y] + 2E[X - E[X]]E[Y - E[Y]]
\]

Since the random variables \( X - E[X] \) and \( Y - E[Y] \) are independent.

But \( E[X - E[X]] = E[X] - E[X] = 0 \).

\( \square \)
Let $X$ be a 0-1 random variable such that
\[
Pr(X = 1) = p, \quad Pr(X = 0) = 1 - p.
\]

\[
E[X] = 1 \cdot p + 0 \cdot (1 - p) = p.
\]

\[
\]
A Binomial Random variable

Consider a sequence of $n$ independent Bernoulli trials $X_1, \ldots, X_n$. Let

$$X = \sum_{i=1}^{n} X_i.$$ 

$X$ has a Binomial distribution $X \sim B(n, p)$.

$$Pr(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}.$$ 

$$E[X] = np.$$ 

$$Var[X] = np(1 - p).$$
Assume again that we flip $N$ coins. Let $X$ be the number of heads. $X_i = 1$ if the $i$-th flip was a head else $X_i = 0$. $E[X_i] = 1/2$. $Var[X_i] = 1/4$.

$$Pr(X \geq 3N/4) \leq Pr(|X - E[X]| \geq N/4) =$$

$$Pr(|X - E[X]| \geq E[X]/2) \leq \frac{Var[X]}{(E[X])^2(1/4)} =$$

$$\frac{N/4}{(N^2/4)(1/4)} = \frac{N/4}{N^2/4} = 4/N.$$  

A significantly better bound than 2/3.
The Advantage of Multiple Samples

**Theorem**

*For any random variable* $X$ *and constant* $a$,

$$\text{Var}[aX] = a^2 \text{Var}[X].$$

**Proof.**

$$\text{Var}[aX] = E[(aX - E[aX])^2] = E[a^2(X - E[X])^2] = a^2 E[(X - E[X])^2] = a^2 \text{Var}[X].$$
Theorem

Let $X_1, \ldots, X_n$ be $n$ independent, identically distributed random variables. Let $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$.

$$\text{Var}[\bar{X}] = \text{Var} \left[ \frac{1}{n} \sum_{i=1}^{n} X_i \right] = \frac{1}{n^2} \text{Var} \left[ \sum_{i=1}^{n} X_i \right] = \frac{1}{n} \text{Var}[X_i].$$
The (Weak) Law of Large Numbers

**Theorem**

Let $X_1, \ldots, X_n$ be independent, identically distributed, random variables. Let $\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$. For any constant $\varepsilon > 0$,

$$\lim_{n \to \infty} \Pr(|\bar{X}_n - \mathbb{E}[X_i]| \leq \varepsilon) = 1.$$ 

**Proof.**

$\text{Var}[\bar{X}_n] = \frac{1}{n} \text{Var}[X_i]$. Applying Chebyshev's inequality

$$\Pr(|\bar{X}_n - \mathbb{E}[X_i]| > \varepsilon) \leq \frac{\text{Var}[X_i]}{n\varepsilon^2}.$$ 

[Can be proven even when $\text{Var}[X_i]$ is not bounded.]