Problem 1

a. We show $Y_i$ is a martingale with respect to $X_i$ and is therefore a martingale with respect to itself. It is clear that $Y_i$ is a function of $X_1, X_2, \ldots, X_i$.

We now show that $E(|Y_i|) < \infty$.

$$E(|Y_i|) = E(| \sum_{j=1}^{i} (X_j - E(X_j|X_1, \ldots, X_{j-1})) |) = E(\sum_{j=1}^{i} X_j - \sum_{j=1}^{i} E(X_j|X_1, \ldots, X_{j-1})) $$

$$\leq E(\sum_{j=1}^{i} X_j) + E(\sum_{j=1}^{i} E(X_j|X_1, \ldots, X_{j-1})) $$

$$\leq 2 \cdot n \cdot i < \infty$$

Where the final equality and the final inequality use the fact that $0 \leq X_j \leq n$ for all $j$.

Finally, we show that $E(Y_{m+1}|X_1, \ldots, X_m) = Y_m$.

$$E(Y_{m+1}|X_1, \ldots, X_m) = E(\sum_{j=1}^{m+1} (X_j - E(X_j|X_1, \ldots, X_{j-1}))|X_1, \ldots, X_m) $$

$$= E(Y_m + X_{m+1} - E(X_{m+1}|X_1, \ldots, X_m)|X_1, \ldots, X_m) $$

$$= Y_m + E(X_{m+1}|X_1, \ldots, X_m) - E(E(X_{m+1}|X_1, \ldots, X_m)|X_1, \ldots, X_m) $$

Which, by $E(V|W) = E(E(V|U,W)|W)$, equals

$$Y_m + E(E(X_{m+1}|X_1, \ldots, X_m, X_1, \ldots, X_m)|X_1, \ldots, X_m) - E(E(X_{m+1}|X_1, \ldots, X_m)|X_1, \ldots, X_m) $$

$$= Y_m + E(X_{m+1}|X_1, \ldots, X_m)|X_1, \ldots, X_m) - E(E(X_{m+1}|X_1, \ldots, X_m)|X_1, \ldots, X_m) = Y_m $$

b. We first claim that $E(X_j|X_1, \ldots, X_{j-1}) = 1$ for all $1 \leq j \leq R$.

Since $j \leq R$, then $\sum_{i=1}^{j-1} X_i < n$, and there are $k > 0$ keys remaining at the start of the $j$th round. The value of $E(X_j|X_1, \ldots, X_{j-1})$ depends only on the number of keys that remain. That is $E(X_j|X_1, \ldots, X_{j-1}) = E(X_j|k \text{ keys remain})$.

For $1 \leq i \leq k$, let $Z_i$ be the 0-1 indicator variable for whether the $i$th remaining customer gets the correct key in round $j$. $E(Z_i) = 1 \cdot \frac{(k-1)!}{k!} + 0 \cdot \frac{k!(k-1)!}{k!} = \frac{1}{k}$.

$E(X_j|k \text{ keys remain}) = E(\sum_{i=1}^{k} Z_i) = \sum_{i=1}^{k} E(Z_i) = k \cdot \frac{1}{k} = 1$, where the second equality uses linearity of expectation.

To use the martingale stopping theorem, we first show that $E(R) < \infty$. In every round, there is a probability $p \geq \frac{1}{n!}$ of returning all keys correctly and thus ending the process. The expected
number of rounds is therefore at most the expected value of the geometric distribution with probability $\frac{1}{n!}$, which is $n! < \infty$.

We now show that $E(|Y_{i+1} - Y_i| | X_1, ..., X_i)$ is bounded by a constant.

$$E(|Y_{i+1} - Y_i| | X_1, ..., X_i) = E(|X_{i+1} - E(X_{i+1} | X_1, ..., X_i)| | X_1, ..., X_i)$$

$$\leq E(|X_{i+1}| + |E(X_{i+1} | X_1, ..., X_i)| | X_1, ..., X_i)$$

$$= E(X_{i+1} | X_1, ..., X_i) + E(E(X_{i+1} | X_1, ..., X_i) | X_1, ..., X_i) \leq 2n$$

Where again, we use the fact that $0 \leq X_j \leq n$ for all $j$.

We have satisfied the conditions for the stopping theorem, so we can conclude that $E(Y_R) = E(Y_1) = E(X_1) = 0$. That is,

$$E(\sum_{j=1}^R X_j - E(\sum_{j=1}^R X_j | X_1, ..., X_{j-1})) = 0$$

$$E(\sum_{j=1}^R X_j - \sum_{j=1}^R E(X_j | X_1, ..., X_{j-1})) = 0$$

$$n - E(\sum_{j=1}^R E(X_j | X_1, ..., X_{j-1})) = 0$$

By the above claim, $E(X_j | X_1, ..., X_{j-1}) = 1$ for all $j$, giving

$$n - E(R \cdot 1) = 0$$

$$E(R) = n$$

**Problem 2**

a. Each vertex has $n - 1$ possible incident edges, so the probability that a particular vertex is isolated is $(1 - p)^{n-1}$. If random variable $X_i$ equals 1 if vertex $i$ is isolated and 0 otherwise, then $E(X_i) = (1 - p)^{n-1}$, and by linearity of expectation, $E(X) = E(\sum_{i=1}^n X_i) = n(1 - p)^{n-1}$.

b. Number the $\binom{n}{2}$ possible edges in some order, and for $i = 1, \ldots, \binom{n}{2}$, let random variable $Y_i$ equal 1 if edge $i$ is present in the graph and 0 otherwise. Let $X = f(Y_1, \ldots, Y_{\binom{n}{2}})$ be the number of isolated vertices if the edges are given by $Y_1, \ldots, Y_{\binom{n}{2}}$. Because adding a single edge decreases the number of isolated vertices by at most 2 and removing a single edge increases the number of isolated vertices by at most 2, $f$ satisfies the Lipschitz condition with bound 2. Since the $Y_i$ are independent, we can apply McDiarmid’s inequality (Theorem 13.7) to obtain

$$\Pr(|X - E(X)| \geq \lambda) \leq 2e^{-2\lambda^2/(\binom{n}{2}2^2)} = 2e^{-\lambda^2/(n(n-1))}.$$
Problem 3

Let $X_1$ be a random variable that takes value 0 with probability $1/2$ and 1 with probability $1/2$. Let $X_2 = X_3 = \ldots = X_n = X_1$. Let

$$f(X_1, \ldots, X_n) = \sum_{i=1}^{n} X_i.$$ 

Because each $X_i$ is a 0/1-random variable, $f$ satisfies the Lipschitz condition with Lipschitz constant $c = 1$. Because $f(X_1, \ldots, X_n)$ takes value 0 with probability $1/2$ and value $n$ with probability $1/2$, we have $Z_0 = E(f(X_1, \ldots, X_n)) = n/2$. But $Z_1$ is either 0 or $n$, because all the $X_i$ are determined by the value of $X_1$. So, for $n > 2$, we have $|Z_1 - Z_0| = n/2 > c$. 