Problem 1

We know that \( E(|Z_i|) < \infty \) because \( Z_0, \ldots, Z_n \) is a martingale with respect to \( X_0, \ldots, X_n \). So, we need only show that

\[
E(Z_{n+1} \mid Z_0, \ldots, Z_n) = Z_n.
\]

Indeed,

\[
E(Z_{n+1} \mid Z_0, \ldots, Z_n) = E(E(Z_{n+1} \mid X_0, \ldots, X_n, Z_0, \ldots, Z_n) \mid Z_0, \ldots, Z_n)
\]

(1)

\[
= E(E(Z_{n+1} \mid X_0, \ldots, X_n) \mid Z_0, \ldots, Z_n)
\]

(2)

\[
= E(Z_n \mid Z_0, \ldots, Z_n)
\]

(3)

\[
= Z_n,
\]

(4)

where (1) holds because \( E(V \mid W) = E(E(V \mid U,W) \mid W) \); (2) holds because \( Z_0, \ldots, Z_n \) is a function of \( X_0, \ldots, X_n \); and (3) holds because \( Z_0, \ldots Z_n \) is a martingale with respect to \( X_0, \ldots, X_n \).

Problem 2

We will use the following facts:

1. For \( i \neq j \), \( E(X_i X_j) = 0 \). This is because \( X_i \) and \( X_j \) are independent, so \( E(X_i X_j) = E(X_i)E(X_j) = (0)(0) = 0 \).

2. \( \sigma^2 = E(X_i^2) \). This is because \( V(X_i) = E(X_i^2) - E(X_i)^2 = E(X_i^2) - 0 \).

We will show that \( E(|Z_n|) \) is finite. Define \( Y_n = (\sum_{i=1}^n X_i)^2 \). We start by showing that \( E(Y_n) \) is finite. By linearity of expectation, \( E(Y_n) = \sum_{i=1}^n E(X_i^2) + 2 \sum_{i \neq j} E(X_i X_j) \). But the first fact shows that the first sum is finite, and the second fact shows that the second sum is 0. Therefore, \( E(Y_n) \) is finite. Now, we show that \( E(|Z_n|) = E(|Y_n - n\sigma^2|) \) is finite. By the triangle inequality, \( |Y_n - n\sigma^2| \leq |Y_n| + |n\sigma^2| \). So, \( E(|Y_n - n\sigma^2|) \leq E(|Y_n|) + E(|n\sigma^2|) \). We just showed that the first term is finite (note that \( Y_n = |Y_n| \)), and the second term is also finite. Therefore, \( E(|Z_n|) \) is finite.

Now, we show that \( E(Z_{n+1} \mid X_1, \ldots, X_n) = Z_n \). We have

\[
Z_{n+1} = \left( \sum_{i=1}^{n+1} X_i \right)^2 - (n+1)\sigma^2
\]

\[
= \left( \sum_{i=1}^{n} X_i \right)^2 - n\sigma^2 + 2 \left( \sum_{i=1}^{n} X_i X_{n+1} \right) + X_{n+1}^2 - \sigma^2
\]

\[
= Z_n + 2 \left( \sum_{i=1}^{n} X_i X_{n+1} \right) + X_{n+1}^2 - \sigma^2.
\]
So, by using linearity of expectation and the fact that \( X_{n+1} \) is independent of \( X_1, \ldots, X_n \),

\[
E(Z_{n+1} \mid X_1, \ldots, X_n) = E\left(Z_n + 2 \left( \sum_{i=1}^{n} X_i X_{n+1} \right) + X_{n+1}^2 - \sigma^2 \mid X_1, \ldots, X_n \right) \\
= Z_n + 2 \left( \sum_{i=1}^{n} X_i E(X_{n+1}) \right) + E(X_{n+1}^2) - \sigma^2 \\
= Z_n + E(X_{n+1}^2) - \sigma^2 \quad \text{(because } E(X_{n+1}) = 0) \\
= Z_n. \quad \text{(by the second fact)}
\]

This shows that \( Z_1, Z_2, \ldots \) is a martingale with respect to \( X_1, X_2, \ldots \) (and also a martingale with respect to itself, due to problem 1).

**Problem 3**

a. This follows from problem 2, because \( V(X_i) = 1 \).

b. We will use the third condition in the martingale stopping theorem. The second part of the condition holds because \( \left( \sum_{i=1}^{n} X_i \right)^2 \) is bounded (between 0 and \( \max(l_1^2, l_2^2) \)), so we need only show that \( E(T) \) is finite. Let \( C = (l_1 + l_2)/2 \). For each \( i, -l_1 < i < l_2 \), let \( p_i \) be the probability that, if the player currently has \( i \) dollars, the game ends within \( C \) steps. By choice of \( C \), all \( p_i \) are nonzero. Let \( p = \min p_i \), so the player always has probability at least \( p \) of ending within the next \( C \) steps. Now, from the beginning of the game, with probability at least \( p \), we end within \( C \) steps. If we haven’t finished yet, with probability at least \( p \), we end within a further \( C \) steps, and so on. Therefore, the expected number of “sequences of \( C \) steps” until the game ends is at most the expectation of a geometric random variable with probability \( p \), which is finite. (A “sequence of \( C \) steps” means steps 1 through \( C \), steps \( C + 1 \) through \( 2C \), and so on.) Therefore, \( E(T) \) is finite, and the martingale stopping theorem tells us that \( E(Z_T) = 0 \).

c. We have

\[
Z_T = \left( \sum_{i=1}^{T} X_i \right)^2 - T,
\]

and using part b and linearity of expectation,

\[
0 = E(Z_T) = E(\left( \sum_{i=1}^{T} X_i \right)^2) - E(T).
\]
Using the result that the probability of winning is $l_1/(l_1 + l_2)$,

$$E(T) = E((\sum_{i=1}^T X_i)^2)$$

$$= \frac{l_1}{l_1 + l_2} l_2^2 + \frac{l_2}{l_1 + l_2} (-l_1)^2$$

$$= \frac{l_1l_2(l_1 + l_2)}{l_1 + l_2}$$

$$= l_1l_2.$$