Homework 6

Due: Apr 11, 2019 at 2:30pm

Please follow these guidelines when writing up your CS 155 homeworks.
A major goal of this class is that you learn how to mathematically analyze probabilistic processes and events. Hence, make sure your proofs are rigorous, i.e., that every step is adequately justified. Try to keep your proofs as simple and as concise as possible, keeping in mind that you will potentially have to examine multiple proof strategies in order to achieve this. Make sure to CLEARLY STATE YOUR ASSUMPTIONS at the beginning of any solution, when necessary. As we are doing probabilistic analysis, we shall require that you clearly define the probability space you are working with and identify the events within it that you will analyze.

You are allowed to discuss these problems with other students, but you have to leave any meeting with other students without any written material about the problems, and the writeups must be done by yourself without help from others. If you have any questions regarding how these guidelines apply to a particular problem or what they mean in general, please email the TAs.

Problem 1
Exercise 13.1 in the textbook

Problem text:
Show that if $Z_0, Z_1, \ldots, Z_n$ is a martingale with respect to $X_0, X_1, \ldots, X_n$, then it is also a martingale with respect to itself.

Problem 2
Exercise 13.4 in the textbook

Problem text:
Let $X_1, X_2, \ldots$ be independent and identically distributed random variables with expectation 0 and variance $\sigma^2 < \infty$. Let
\[
Z_n = \left( \sum_{i=1}^{n} X_i \right)^2 - n\sigma^2.
\]
Show that $Z_1, Z_2, \ldots$ is a martingale.

Problem 3

Exercise 13.5 in the textbook

Problem text:

Consider the gambler’s ruin problem, where a player plays a sequence of independent games, either winning one dollar with probability $1/2$ or losing one dollar with probability $1/2$. The player continues until either losing $l_1$ dollars or winning $l_2$ dollars. Let $X_n$ be 1 if the player wins the $n$th game and $-1$ otherwise. Let $Z_n = \left(\sum_{i=1}^{n} X_i\right)^2 - n$.

a. Show that $Z_1, Z_2, \ldots$ is a martingale.

b. Let $T$ be the stopping time when the player finishes playing. Determine $E(Z_T)$.

c. Calculate $E(T)$. (Hint: you can use what you already know about the probability that the player wins.)