Problem 1

(Exercise 4.6)

(a) In an election with two candidates using paper ballots, each vote is independently misrecorded with probability $p = 0.02$. Use a Chernoff bound to bound the probability that more than 4% of the votes are misrecorded in an election of 1,000,000 ballots.

(Solution)

For $i = 1, 2, \ldots, 1,000,000$, let $X_i$ be the random variable defined by following:

$$X_i = \begin{cases} 0, & \text{if the } i\text{th ballot was not misrecorded} \\ 1, & \text{Otherwise} \end{cases}$$

Also, for convenience, let $N = 1,000,000$.

Then, we can consider each $X_i$ as the Bernoulli $(0, 1)$ random variable such that $\mathbb{P}(X_i = 1) = p_i = p = 0.02$.

Also, $(X_i)_{i=1}^N$ are independent and what we hope to bound is the probability $\mathbb{P}\left(\sum_{i=1}^N X_i > 0.04N\right)$.

Note that, for $X = \sum_{i=1}^N X_i$, $\mu = \sum_{i=1}^N p_i = 0.02N$ and $\delta = 1$, by the Chernoff bound, the following holds:

$$\mathbb{P}\left(\sum_{i=1}^N X_i > 0.04N\right) \leq \mathbb{P}(X \geq (1 + \delta)\mu) \leq \exp\left(-\mu \delta^2/3\right) = \exp\left(-\frac{0.02N}{3}\right) = \exp\left(-\frac{20000}{3}\right) \approx 10^{-2895.30}$$

I.e., the probability that more than 4% of the votes are misrecorded in an election of 1,000,000 ballots is bounded by $\exp(-20000/3) \approx 10^{-2895.30}$.

If we use another form of the Chernoff bound $\mathbb{P}(X \geq (1 + \delta)\mu) \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\mu$ instead, we can get a better bound:

$$\mathbb{P}\left(\sum_{i=1}^N X_i > 0.04N\right) \leq \mathbb{P}(X \geq (1 + \delta)\mu) \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\mu = \left(\frac{e^{0.02}}{2^2}\right)^{0.02N} = \left(\frac{e^{0.02}}{4}\right)^{20000} \approx 10^{-3355.31}$$

(b) Assume that a misrecorded ballot always counts as a vote for the other candidate. Suppose that candidate A received 510,000 votes and that candidate B received 490,000 votes. Use Chernoff bounds to bound the probability that candidate B wins the election owing to misrecorded ballots. Specifically, let $X$ be the number of votes for candidate A that are misrecorded and let $Y$ be the number of votes for candidate B that are misrecorded. Bound $\mathbb{P}\left((X > k) \land (Y < l)\right)$ for suitable choices of $k$ and $l$.

(Solution)

Let $X$ be the number of votes for candidate A that are misrecorded and let $Y$ be the number of votes for candidate B that are misrecorded. Then, the event that candidate B wins if and only if $510000 - X + Y < 490000 + X - Y$, which is equivalent to $10000 + Y < X$. 

CSCI2540 - Homework 2

Homework 2

Problem 1

(Exercise 4.6)

(a) In an election with two candidates using paper ballots, each vote is independently misrecorded with probability $p = 0.02$. Use a Chernoff bound to bound the probability that more than 4% of the votes are misrecorded in an election of 1,000,000 ballots.

(Solution)

For $i = 1, 2, \ldots, 1,000,000$, let $X_i$ be the random variable defined by following:

$$X_i = \begin{cases} 0, & \text{if the } i\text{th ballot was not misrecorded} \\ 1, & \text{Otherwise} \end{cases}$$

Also, for convenience, let $N = 1,000,000$.

Then, we can consider each $X_i$ as the Bernoulli $(0, 1)$ random variable such that $\mathbb{P}(X_i = 1) = p_i = p = 0.02$.

Also, $(X_i)_{i=1}^N$ are independent and what we hope to bound is the probability $\mathbb{P}\left(\sum_{i=1}^N X_i > 0.04N\right)$.

Note that, for $X = \sum_{i=1}^N X_i$, $\mu = \sum_{i=1}^N p_i = 0.02N$ and $\delta = 1$, by the Chernoff bound, the following holds:

$$\mathbb{P}\left(\sum_{i=1}^N X_i > 0.04N\right) \leq \mathbb{P}(X \geq (1 + \delta)\mu) \leq \exp\left(-\mu \delta^2/3\right) = \exp\left(-\frac{0.02N}{3}\right) = \exp\left(-\frac{20000}{3}\right) \approx 10^{-2895.30}$$

I.e., the probability that more than 4% of the votes are misrecorded in an election of 1,000,000 ballots is bounded by $\exp(-20000/3) \approx 10^{-2895.30}$.

If we use another form of the Chernoff bound $\mathbb{P}(X \geq (1 + \delta)\mu) \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\mu$ instead, we can get a better bound:

$$\mathbb{P}\left(\sum_{i=1}^N X_i > 0.04N\right) \leq \mathbb{P}(X \geq (1 + \delta)\mu) \leq \left(\frac{e^\delta}{(1+\delta)^{1+\delta}}\right)^\mu = \left(\frac{e^{0.02}}{2^2}\right)^{0.02N} = \left(\frac{e^{0.02}}{4}\right)^{20000} \approx 10^{-3355.31}$$

(b) Assume that a misrecorded ballot always counts as a vote for the other candidate. Suppose that candidate A received 510,000 votes and that candidate B received 490,000 votes. Use Chernoff bounds to bound the probability that candidate B wins the election owing to misrecorded ballots. Specifically, let $X$ be the number of votes for candidate A that are misrecorded and let $Y$ be the number of votes for candidate B that are misrecorded. Bound $\mathbb{P}\left((X > k) \land (Y < l)\right)$ for suitable choices of $k$ and $l$.

(Solution)

Let $X$ be the number of votes for candidate A that are misrecorded and let $Y$ be the number of votes for candidate B that are misrecorded. Then, the event that candidate B wins if and only if $510000 - X + Y < 490000 + X - Y$, which is equivalent to $10000 + Y < X$. 

Note that, because $0 \leq X \leq 510000$ and $0 \leq Y \leq 490000$, for any $0 \leq l \leq 490000$, the following holds:

\[
\mathbb{P}(10000 + Y < X) = \mathbb{P}((10000 + Y < X) \cap (0 \leq Y \leq 490000)) \\
= \mathbb{P}((10000 + Y < X) \cap (0 \leq Y \leq l) \cup (l < Y \leq 490000)) \\
= \mathbb{P}((10000 + Y < X) \cap (0 \leq Y \leq l)) + \mathbb{P}((10000 + Y < X) \cap (l < Y \leq 490000)) \\
\leq \mathbb{P}(0 \leq Y \leq l) + \mathbb{P}(10000 + l < X) \leq \mathbb{P}(Y \leq l) + \mathbb{P}(10000 + l \leq X)
\]

For $i = 1, 2, \ldots, 510000$ and $j = 1, 2, \ldots, 490000$, let $X_i$ and $Y_j$ be the random variables defined by following:

\[
X_i = \begin{cases} 
0, & \text{if the } i\text{th ballot for candidate A was not misrecorded} \\
1, & \text{Otherwise}
\end{cases}
\]

\[
Y_j = \begin{cases} 
0, & \text{if the } j\text{th ballot for candidate B was not misrecorded} \\
1, & \text{Otherwise}
\end{cases}
\]

Then, by the definitions, $X = \sum_{i=1}^{510000} X_i$ and $Y = \sum_{j=1}^{490000} Y_j$ and $\{X_i\}_{i=1}^{510000}$, $\{Y_j\}_{j=1}^{490000}$ are independent.

Let $\mu_X = \sum_{i=1}^{510000} p_i = 0.02 \times 510000 = 10200$, $\mu_Y = \sum_{j=1}^{490000} p_j = 0.02 \times 490000 = 9800$, and $\delta = 0.48$.

Then, by the Chernoff bound, the following inequalities hold:

\[
\mathbb{P}(X \geq 10000 + 5096) = \mathbb{P}(X \geq (1 + 0.48) \times 10200) = \mathbb{P}(X \geq (1 + \delta) \mu_X) \\
\leq \exp(-\frac{\mu_X \delta^2}{3}) = \exp(-\frac{10200 \times 0.48^2}{3}) = \exp(-783.36) \approx 10^{-340.2089}
\]

\[
\mathbb{P}(Y \leq 5096) = \mathbb{P}(Y \leq (1 - 0.48) \times 9800) = \mathbb{P}(Y \leq (1 - \delta) \mu_Y) \\
\leq \exp(-\frac{\mu_Y \delta^2}{2}) = \exp(-\frac{9800 \times 0.48^2}{2}) = \exp(-1128.96) \approx 10^{-490.3011}
\]

I.e., if we take $l = 5096$, we can give an upper bound for the probability that candidate B wins the election owing to misrecorded ballots by following:

\[
\mathbb{P}(10000 + Y < X) \leq \mathbb{P}(Y \leq 5096) + \mathbb{P}(10000 + 5096 \leq X) \\
\leq \exp(-783.36) + \exp(-1128.96) \approx 10^{-340.2089} + 10^{-490.3011}
\]

If we use another form of the Chernoff bound $\mathbb{P}(X \geq (1 + \delta) \mu_X) \leq \left(e^{\frac{\delta}{1+\delta}}\right)^{\mu_X}$ and $\mathbb{P}(Y \leq (1 - \delta) \mu_Y) \leq \left(e^{-\delta}\right)^{\mu_Y}$ instead, we can get a better bound:

\[
\mathbb{P}(X \geq 10000 + 5096) = \mathbb{P}(X \geq (1 + \delta) \mu_X) \leq \left(e^{\frac{\delta}{1+\delta}}\right)^{\mu_X} = \left(e^{\frac{0.48}{1+0.48}}\right)^{10200} \approx 10^{-443.9651}
\]

\[
\mathbb{P}(Y \leq 5096) = \mathbb{P}(Y \leq (1 - \delta) \mu_Y) \leq \left(e^{-\delta}\right)^{\mu_Y} = \left(e^{-0.48}\right)^{9800} \approx 10^{-595.6743}
\]

\[
\Rightarrow \mathbb{P}(10000 + Y < X) \leq \left(e^{\frac{0.48}{1+0.48}}\right)^{10200} + \left(e^{-0.48}\right)^{9800} \approx 10^{-443.9651} + 10^{-595.6743}
\]
Suppose that we can obtain independent samples $X_1, X_2, \cdots$ of a random variable $X$ and that we want to use these samples to estimate $\mathbb{E}[X]$. Using $t$ samples, we use $\frac{\sum_{i=1}^{t} X_i}{t}$ for our estimate of $\mathbb{E}[X]$. We want the estimate to be within $\varepsilon \mathbb{E}[X]$ from the true value of $\mathbb{E}[X]$ with probability at least $1 - \delta$. We may not be able to use Chernoff bound directly to bound how good our estimate is if $X$ is not a 0-1 random variable, and we do not know its moment generating function. We develop an alternative approach that requires only having a bound on the variance of $X$. Let $r = \sqrt{\text{Var}[X]} / \mathbb{E}[X]$.

(a) Show using Chebyshev’s inequality that $O(r^2 / \varepsilon^2 \delta)$ samples are sufficient to solve the problem.

(Solution)

Let $Y_t \triangleq \frac{\sum_{i=1}^{t} X_i}{t}$. Then $\mathbb{E}[Y_t] = \frac{\mathbb{E}\left[ \sum_{i=1}^{t} X_i \right]}{t} = \frac{t \mathbb{E}[X]}{t} = \mathbb{E}[X]$ and, by the independency, $\text{Var}[Y_t] = \text{Var}\left[ \frac{\sum_{i=1}^{t} X_i}{t} \right] = \frac{\sum_{i=1}^{t} \text{Var}[X_i]}{t^2} = \frac{t \text{Var}[X]}{t^2} = \frac{\text{Var}[X]}{t}$.

Therefore, by using Chebyshev’s inequality on $Y_t$, the following holds:

$$\mathbb{P}\left( \frac{1}{t} \sum_{i=1}^{t} X_i - \mathbb{E}[X] \geq \varepsilon \mathbb{E}[X] \right) = 1 - \mathbb{P}\left( \frac{1}{t} \sum_{i=1}^{t} X_i - \mathbb{E}[X] \geq \varepsilon \mathbb{E}[X] \right) = 1 - \mathbb{P}(Y_t - \mathbb{E}[Y_t] \geq \varepsilon \mathbb{E}[Y_t]) \geq 1 - \frac{\text{Var}[Y_t]}{\varepsilon^2 \mathbb{E}[Y_t]^2} = 1 - \frac{\text{Var}[X]}{t^2 \varepsilon^2 \mathbb{E}[X]^2} = 1 - \frac{1}{t} \cdot \frac{r^2}{\varepsilon^2}$$

I.e., if $t \geq r^2 / \varepsilon^2 \delta$,

$$\mathbb{P}\left( \frac{1}{t} \sum_{i=1}^{t} X_i - \mathbb{E}[X] \geq \varepsilon \mathbb{E}[X] \right) \geq 1 - \frac{1}{t} \cdot \frac{r^2}{\varepsilon^2} \geq 1 - \frac{1}{r^2 / \varepsilon^2 \delta} \cdot \frac{r^2}{\varepsilon^2} = 1 - \delta$$

which shows that $O(r^2 / \varepsilon^2 \delta)$ samples are sufficient to solve the problem.

(b) Suppose that we need only a weak estimate that is within $\varepsilon \mathbb{E}[X]$ of $\mathbb{E}[X]$ with probability at least $3/4$. Argue that $O(r^2 / \varepsilon^2)$ samples are enough for this weak estimate.

(Solution)

By the bound in (a), if $t \geq 4r^2 / \varepsilon^2$, we can get the following:

$$\mathbb{P}\left( \frac{1}{t} \sum_{i=1}^{t} X_i - \mathbb{E}[X] \geq \varepsilon \mathbb{E}[X] \right) \geq 1 - \frac{1}{t} \cdot \frac{r^2}{\varepsilon^2} \geq 1 - \frac{1}{4r^2 / \varepsilon^2} \cdot \frac{r^2}{\varepsilon^2} = 1 - \frac{1}{4} = \frac{3}{4}$$

which shows that $O(r^2 / \varepsilon^2)$ samples are enough for this weak estimate.
Show that, by taking the median of $\mathcal{O}(\log(1/\delta))$ weak estimates, we can obtain an estimate within $\varepsilon \mathbb{E}[X]$ of $\mathbb{E}[X]$ with probability at least $1 - \delta$. Conclude that we need only $\mathcal{O}(r^2 \log(1/\delta)/\varepsilon^2)$ samples.

(Solution)

Suppose that we computed $N$ weak estimates, say, $Y^1, Y^2, \ldots, Y^N$, and define the random variables $Z_n$ by following:

$$Z_n = \begin{cases} 0, & \text{if } |Y^n - \mathbb{E}[X]| \geq \varepsilon \mathbb{E}[X] \\ 1, & \text{Otherwise} \end{cases}$$

Then, by (b), we can say that $p_n \triangleq \mathbb{P}(Z_n = 1) \geq 3/4$ for each $n = 1, 2, \ldots, N$. Let $Z \triangleq \sum_{n=1}^{N} Z_n$ and $\mu \triangleq \sum_{n=1}^{N} p_n$. Let $Y$ be the median of $\{Y^n\}_{n=1}^{N}$. Then $|Y - \mathbb{E}[X]| \geq \varepsilon \mathbb{E}[X]$ implies that at least half of $Y^n$ is outside of the $\varepsilon$-ball of $\mathbb{E}[X]$, which means that at least half of $Z_n$ is 0.

I.e., because $\mu = \sum_{n=1}^{N} p_n \geq 3N/4$ and we can consider that $Z_n$’s are independent, by Chernoff bound, for $\gamma = 1/3$, 

$$\mathbb{P}(|Y - \mathbb{E}[X]| \geq \varepsilon \mathbb{E}[X]) \leq \mathbb{P}\left(\sum_{n=1}^{N} Z_n \leq \frac{N}{2}\right) = \mathbb{P}\left(Z \leq \frac{N}{2}\right) = \mathbb{P}\left(Z \leq \left(1 - \frac{1}{3}\right) \frac{3N}{4}\right) \leq \mathbb{P}\left(Z \leq \left(1 - \frac{1}{3}\right) \mu\right)$$

$$= \mathbb{P}(Z \leq (1 - \gamma)\mu) \leq \exp(-\mu \gamma^2/2) \leq \exp( -(3N/4)\gamma^2/2) = \exp(-N/24)$$

Therefore, if $N \geq 24 \log(1/\delta)$, we have the following bound for the median $Y$:

$$\mathbb{P}(|Y - \mathbb{E}[X]| < \varepsilon \mathbb{E}[X]) = 1 - \mathbb{P}(|Y - \mathbb{E}[X]| \geq \varepsilon \mathbb{E}[X]) \geq 1 - \exp(-N/24) \geq 1 - \delta$$

Because, by (b), $\mathcal{O}(r^2/\varepsilon^2)$ samples are required for computing each weak estimate, we can conclude that we need only $\mathcal{O}(r^2 \log(1/\delta)/\varepsilon^2)$ samples for obtaining an estimate within $\varepsilon \mathbb{E}[X]$ of $\mathbb{E}[X]$ with probability at least $1 - \delta$.

If we use another form of the Chernoff bound $\mathbb{P}(Z \leq (1 - \gamma)\mu) \leq \left(\frac{e^{-\gamma}}{(1-\gamma)^{1-\gamma}}\right)^\mu$ instead, we can get the same result but with more complicated (dirty) expressions, so I skip it in this answer.

\[\square\]
Problem 3

(Exercise 4.18)

In many wireless communication systems, each receiver listens on a specific frequency. The bit \( b(t) \) sent at time \( t \) is represented by 1 or −1. Unfortunately, noise from other nearby communications can affect the receiver’s signal. A simplified model of this noise is as follows. There are \( n \) other senders, and the \( i \)th has strength \( p_i \leq 1 \). At any time \( t \), the \( i \)th sender is also trying to send a bit \( b_i(t) \) that is represented by 1 or −1. The receiver obtains the signal \( s(t) \) given by

\[
s(t) = b(t) + \sum_{i=1}^{n} p_i b_i(t)
\]

If \( s(t) \) is closer to 1 than −1, the receiver assumes that the bit sent at time \( t \) was a 1; otherwise, the receiver assumes that it was a −1.

Assume that all the bits \( b_i(t) \) can be considered independent, uniformly random variables. Give a Chernoff bound to estimate the probability that the receiver makes an error in determining \( b(t) \).

(Solution)

If \( \sum_{i=1}^{n} p_i^2 = 0 \), \( p_i = 0 \) for any \( i \), so \( s(t) \equiv b(t) \), which means the receiver never makes an error in determining \( b(t) \), so we can give any bound on the probability of error.

Suppose that \( \sum_{i=1}^{n} p_i^2 > 0 \). The receiver makes an error in determining \( b(t) \) if and only if either ‘\( b(t) = 1 \) and \( s(t) < 0 \)’ or ‘\( b(t) = -1 \) and \( s(t) > 0 \)’. I.e., we only need to get the bound for the following:

\[
\Pr(\{b(t)=1\} \land \{s(t)<0\}) + \Pr(\{b(t)=-1\} \land \{s(t)>0\})
\]

Note that, by the independence of bits,

\[
\Pr(\{b(t)=1\} \land \{s(t)<0\}) = \Pr(b(t)=1)\Pr(s(t)<0|b(t)=1)
\]

\[
= \Pr(b(t)=1)\sum_{i=1}^{n} \Pr(p_i b_i(t) < -1 | b(t)=1) = \Pr(b(t)=1)\Pr\left(\sum_{i=1}^{n} p_i b_i(t) < -1 \right)
\]

\[
\Pr(\{b(t)=-1\} \land \{s(t)>0\}) = \Pr(b(t)=-1)\Pr(s(t)>0|b(t)=-1)
\]

\[
= \Pr(b(t)=-1)\Pr\left(\sum_{i=1}^{n} p_i b_i(t) > 1 \right) = \Pr(b(t)=-1)\Pr\left(\sum_{i=1}^{n} p_i b_i(t) > 1 \right)
\]

If \( \sum_{i=1}^{n} p_i b_i(t) < -1 \), then for any \( s > 0 \), \( s \sum_{i=1}^{n} p_i b_i(t) < -s \), which is equivalent to \( e^{-s \sum_{i=1}^{n} p_i b_i(t)} > e^s \), holds.

I.e., for any \( s > 0 \), by Markov inequality and independence,

\[
\Pr\left(\sum_{i=1}^{n} p_i b_i(t) < -1 \right) \leq \Pr(e^{-s \sum_{i=1}^{n} p_i b_i(t)} > e^s) \leq \frac{\mathbb{E}\left[e^{-s \sum_{i=1}^{n} p_i b_i(t)}\right]}{e^s} = e^{-s \sum_{i=1}^{n} \mathbb{E}[p_i b_i(t)]}
\]

Note that \( \Pr(b_i(t) \in [-1,1]) = 1 \) and \( \mathbb{E}[b_i(t)] = 0 \). Moreover, \( \mathbb{E}[e^{-s p_i b_i(t)}] = 0.5 e^{-sp_i} + 0.5 e^{sp_i} = \mathbb{E}[e^{s p_i b_i(t)}] \).

Therefore, by Hoeffding’s Lemma, \( \mathbb{E}[e^{-s p_i b_i(t)}] \leq e^{s^2 p_i^2 (1-(-1))^2/8} = e^{s^2 p_i^2 /2} \) holds for any \( i \), \( s \), and \( p_i \).

I.e., for any \( s > 0 \),
\begin{align*}
e^{-s} \prod_{i=1}^{n} \mathbb{E}[e^{-sp_i b_i(t)}] & \leq e^{-s} \prod_{i=1}^{n} \mathbb{E}[e^{sp_i^2/2}] = \exp \left( \frac{\sum_{i=1}^{n} p_i^2}{2} s^2 - s \right) \\
\end{align*}

which implies that

\begin{align*}
\mathbb{P} \left( \sum_{i=1}^{n} p_i b_i(t) < -1 \right) & \leq \inf_{s > 0} \exp \left( \frac{\sum_{i=1}^{n} p_i^2}{2} s^2 - s \right) \\
\end{align*}

Recall that we assumed \( \sum_{i=1}^{n} p_i^2 > 0 \), so we can explicitly compute \( \inf_{s > 0} \exp \left( \frac{\sum_{i=1}^{n} p_i^2}{2} s^2 - s \right) \), which is equal to \( \exp \left( - \frac{1}{2 \sum_{i=1}^{n} p_i^2} \right) \), when \( s = 1/\sum_{i=1}^{n} p_i^2 \), by differentiating \( \frac{\sum_{i=1}^{n} p_i^2}{2} s^2 - s \) with respect to \( s \).

Therefore,

\begin{align*}
\mathbb{P} \left( \sum_{i=1}^{n} p_i b_i(t) < -1 \right) & \leq \exp \left( - \frac{1}{2 \sum_{i=1}^{n} p_i^2} \right) \\
\end{align*}

Also, with the exactly same logic as above, for any \( s > 0 \),

\begin{align*}
\mathbb{P} \left( \sum_{i=1}^{n} p_i b_i(t) > 1 \right) & \leq \mathbb{P} \left( e^s \Sigma_{i=1}^{n} p_i b_i(t) > e^s \right) \leq \mathbb{E} \left[ e^s \Sigma_{i=1}^{n} p_i b_i(t) \right] = e^s \prod_{i=1}^{n} \mathbb{E}[e^{sp_i b_i(t)}] \\
& \leq e^{-s} \prod_{i=1}^{n} e^{sp_i^2/2} = \exp \left( \frac{\sum_{i=1}^{n} p_i^2}{2} s^2 - s \right) \\
\end{align*}

So, we have the same (symmetric) bound in this case:

\begin{align*}
\mathbb{P} \left( \sum_{i=1}^{n} p_i b_i(t) > 1 \right) & \leq \inf_{s > 0} \exp \left( \frac{\sum_{i=1}^{n} p_i^2}{2} s^2 - s \right) = \exp \left( - \frac{1}{2 \sum_{i=1}^{n} p_i^2} \right) \\
\end{align*}

Therefore, the probability that the receiver makes an error in determining \( b(t) \) is bounded above by following:

\begin{align*}
\mathbb{P} \left( (b(t) = 1) \cap (s(t) < 0) \right) + \mathbb{P} \left( (b(t) = -1) \cap (s(t) > 0) \right) \\
= \mathbb{P} (b(t) = 1) \mathbb{P} \left( \sum_{i=1}^{n} p_i b_i(t) < -1 \right) + \mathbb{P} (b(t) = -1) \mathbb{P} \left( \sum_{i=1}^{n} p_i b_i(t) > 1 \right) \\
\leq \mathbb{P} (b(t) = 1) \exp \left( - \frac{1}{2 \sum_{i=1}^{n} p_i^2} \right) + \mathbb{P} (b(t) = -1) \exp \left( - \frac{1}{2 \sum_{i=1}^{n} p_i^2} \right) \\
= \left( \mathbb{P} (b(t) = 1) + \mathbb{P} (b(t) = -1) \right) \exp \left( - \frac{1}{2 \sum_{i=1}^{n} p_i^2} \right) = \exp \left( - \frac{1}{2 \sum_{i=1}^{n} p_i^2} \right) \\
\end{align*}