Notes on Simplex Algorithm

CS 149 Staff

October 18, 2007

Until now, we have represented the problems geometrically, and solved by finding a corner and moving around. Now we learn an algorithm to solve this without drawing a graph, and feasible regions.
Once we have a standard form of LP, we can construct a simplex tableau, which looks like following.

$$\begin{array}{c|c} c^T & 0 \\ \hline A & b \end{array}$$

as in lecture slide 5 on Simplex.

In this phase, we assume the initial tableau represents feasible solution.
(We will later deal with the case that it doesn't)
So, in the lecture slide 8 on Simplex, we see a tableau, which corresponds to a bfs, $x = 0, y = 0$. This is a Simplex tableau representation of the following problem:

$Min - x - 2y$
$- x + y \leq 2$
$x - y \leq 3$
$x + y \leq 5$
$x \leq 4$
$y \leq 3$
$x, y \geq 0$

We can see that the five variables added in front are slack variables to make it standard form. In this case our bfs and the vectors are:

$$\mathbf{x_0} = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 4 \\ 3 \\ 0 \\ 0 \end{bmatrix} \mathbf{y_1} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \mathbf{y_2} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

With the instructions given in the slide 7, we choose a column with negative c^T, which can either be 6th or 7th. In this case we 'arbitrarily' choose 6th column. And

the next procedure is to find $\lambda = min(\frac{3}{1}, \frac{5}{1}, \frac{4}{1}) = 3$. Since it was the second row that gave us minimum, we choose second row. Therefore 6th column goes into the basis, and second column goes out. We now follow the direction of y_1 for length 3. The result after change of basis comes in next slide, slide 9. Now we have another tableau, which corresponds to a bfs, $x = 3, y = 0$.
Now in this tableau, we have bfs and vectors as following:

$$\mathbf{x_0} = \begin{bmatrix} 5 \\ 0 \\ 2 \\ 1 \\ 3 \\ 3 \\ 0 \end{bmatrix} \mathbf{y_1} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \mathbf{y_2} = \begin{bmatrix} 0 \\ 0 \\ -2 \\ -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

Now if we take a look at the top row, c^T, we see five 0's, and two non-zero's. One is 1, and the other is -3, and we can see those values are actually $c^T y_1$ and $c^T y_2$, respectively (here c is the original objective function). That suggests us that following y_1 will increase our objective value, which we don't want.
Now similarly, we choose a column with negative c^T, which is 7th column. That means we put 7th column, which is y, into basis. We choose $\lambda = min(\frac{2}{2}, \frac{1}{1}, \frac{3}{1}) = 1$. Now we have two rows that have minimum ratio. We choose, again 'arbitrarily', the 4th row, and we will have a new basis of $\{s1, s3, s5, x, y\}$ (slide 10).
Now we have x_0, y_1, y_2 as following:

$$\mathbf{x_0} = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 2 \\ 4 \\ 1 \end{bmatrix} \mathbf{y_1} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \mathbf{y_2} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

Again, we choose a column with negative c^T, which is 2nd column. However, when we draw the original feasible region, and the corner we're in, we can see that y_1 points up, and there is no way to go up from the corner $x = 4, y = 1$. If we calculate $\lambda = min(\frac{5}{1}, \frac{0}{1}, \frac{2}{1}) = 0$, we can see that we're moving 0 on that direction. That is, we are not moving but we are changing our basis. Now s_2 goes in the basis and s_3 goes out.
We do this procedure one more time(slide 11) and we will get a tableau that we have optimal solution(slide 12). As we can see in the top row, there is no negative values, and that means we are in optimal point. And the top-right corner, 8, is the negative of our objective value on this bfs. we can see that $x = 2, y = 3$ will give $-x - 2y = -8$. From this bfs, in which direction we move, either y_1 or y_2, we can't

decrease our objective value.

$$\mathbf{x_0} = \begin{bmatrix} 1 \\ 4 \\ 0 \\ 2 \\ 0 \\ 2 \\ 3 \end{bmatrix} \ \mathbf{y_1} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \ \mathbf{y_2} = \begin{bmatrix} 2 \\ -2 \\ 0 \\ -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

We can see that $c^T y_1 = (-1) * (-1) + 0 * (-2) = 1 \geq 0$ and $c^T y_2 = 1 * (-1) + (-1) * (-2) = 1 \geq 0$, and that means the point (2,3) is the best one in the cone starting from that point with vectors y_1 and y_2, in which whole feasible region dwells. Therefore the point is global minimum.

Now we have a question. Does this algorithm always terminate?

- In case of no degeneracy the objective function will increase strictly monotonically. Therefore, since there are finite number of corners, and we never visit the same corner twice, this algorithm will terminate.
- In case of degeneracy (e.g., LP with $Ax = 0$), the algorithm may cycle forever.

To avoid cycles, we use **Bland's Anticycling Algorithm**. When selecting a pivot:

- choose the lowest numbered (i.e., leftmost) column t with a negative cost $\leftarrow$ entering column
- among rows k with the ratio $\min_{k|\bar{a}_k^t > 0} \frac{\bar{b}_k}{\bar{a}_k^t}$ choose the one with the lowest numbered $B(k) \leftarrow$ leaving column

First, if we have two or more columns that have negative c^T, we choose the leftmost column to enter the basis. Second, if we have two or more rows that have minimal λ, we choose the row where the corresponding column that will be leaving the basis is leftmost.

We are now going to show that Bland's algorithm works.

Suppose we use Bland's algorithm and enter a cycle. Remove from the tableau the rows and columns that do not contain pivots during the cycle. Let T_1 be the tableau before the last column n enters the basis.

$$T_1 : \quad \begin{array}{c|c|c} & n & \\ \geq 0 & < 0 & \\ \hline & & 0 \\ & & \vdots \\ & & 0 \end{array}$$

We denote entries in T_1 with $\bar{c}, \bar{a}$. They satisfy the following:

$$\forall i < n, \bar{c}_i \geq 0 \tag{1}$$
$$\bar{c}_n < 0 \tag{2}$$

Equations 1 and 2 follow from the fact that the lowest numbered column with a negative cost is the last column.

T_2 is the tableau before column n leaves the basis and column p enters the basis.

	p		n	
	< 0		$= 0$	
T_2 :	≤ 0			0
	$\vdots$			$\vdots$
	> 0		1	$0 \leftarrow$ row r
	≤ 0			0

We denote entries in T_2 with $\hat{c}, \hat{a}$. They satisfy the following:

$$\hat{c}_p < 0, \hat{c}_n = 0, \hat{a}_{r,p} > 0 \tag{3}$$
$$\forall i \neq r, \hat{a}_{i,p} \leq 0 \tag{4}$$

The reason why $\forall i \neq r, \hat{a}_{i,p} \leq 0$ is because if $\hat{a}_{i,p} > 0$, its corresponding basic column will be less than n, and by Bland's algorithm, we wouldn't introduce column n into basis. From T_2 we define a solution y by

$$y_j = \begin{cases} -\hat{a}_{i,p} & \text{if } \hat{B}(i) = j \\ 1 & \text{if } j = p \\ 0 & \text{otherwise} \end{cases}$$

You can check that y is indeed a solution to $Ay = b$. Note however that it's neither basic, nor feasible.

We now focus on $\bar{c}^T y$.

(a) It can be shown that $\bar{c}^T y = \hat{c}_p < 0$ (by (3))

(b)

$$\begin{aligned} \bar{c}^T y &= \sum_{i=1}^{n} \bar{c}_i^T y_i \\ &= \sum_{i=1}^{m} -\hat{a}_{i,p} \bar{c}_{\hat{B}(i)} + \bar{c}_p \\ &= \sum_{i=1 i \neq r}^{m} -\hat{a}_{i,p} \bar{c}_{\hat{B}(i)} - \hat{a}_{r,p} \bar{c}_{\hat{B}(r)} + \bar{c}_p \\ &> 0 \end{aligned}$$

$$\because \left(\begin{array}{l} \hat{a}_{i,p} \leq 0 \text{ by } (4) \\ \bar{c}_{\hat{B}(i)} \geq 0 \text{ by } (1) \\ \hat{a}_{r,p} > 0 \text{ by } (3) \\ \bar{c}_{\hat{B}(r)} = \bar{c}_n < 0 \text{ by } (2) \\ \bar{c}_p > 0 \text{ by } (1) \end{array} \right)$$

By (a) and (b), we have a contradiction, and n cannot go back into the basis. Therefore, using Bland's anticycling algorithm, we will not have cycles.