1. **Problems with Dice**: Suppose I roll a 4 sided die and an 8 sided die, each marked with consecutively increasing integers, starting from 1.

   (a) What is the expectation and variance of the 4 and 8 sided die rolls?
   
   (b) What is the expectation and variance of the sum of the two die rolls?
   
   (c) Suppose I flip a fair coin, if heads I roll the 4 sided die, and if tails, I roll the 8 sided die. What are the expectation and variance of the roll value?
   
   (d) What is the probability that the 8-sided die is at least 7 and the 4-sided die is even?
   
   (e) What is the probability that the 8-sided die is at least 7 or the 4-sided die is even?
   
   (f) Suppose I observe that the 8-sided die is at least 7, and the 4-sided die is even. What is the probability that their sum is 9?

2. **Independence**: Suppose $X, Y$ are independent.

   (a) What is $P(X = x \land Y = y)$
   
   (b) What is $P(X = x \lor Y = y)$
   
   (c) Prove that $P(X = x) + P(Y = y) - P(X = x \land Y = y) = 1 - P(X \neq x \land Y \neq y)$. How else could we describe this quantity?
   
   (d) **Bonus**: Is it always the case that $\text{Cov}(X, Y) = 0$? Either prove the statement or demonstrate that it is false with a counterexample.

3. **Conditional Exponentials and Conditional Expectations**: Suppose $\mathcal{X}$ is exponentially distributed with rate parameter $\lambda$.

   (a) Suppose we observe that $\mathcal{X} > \alpha$ for some $\alpha > 0$. Given this observation, how is $\mathcal{X}$ distributed?
   
   (b) What is $E[\mathcal{X} | \mathcal{X} > \alpha]$ for some $\alpha > 0$. Try to use the previous result to solve this problem.
   
   (c) Solve b again, this time using only integration and calculus (Hint: use integration by parts). Do your answers agree? Which strategy do you prefer?
   
   (d) **Fun with Probability Density Functions**: Suppose I flip unbiased coins until achieving the first heads. Then for each coin I flipped, I draw an exponential random variable and sum them. What is the expectation of their sum?
4. **Expectation and Variance**: Suppose $X$ and $Y$ are normally distributed with mean and variance $\mu_X, \sigma^2_X$ and $\mu_Y, \sigma^2_Y$, respectively, and let $\sigma_{X,Y} = \text{Cov}(X,Y)$.

**Hint**: Recall that $\mathbb{V}[aX + bY] = a^2 \mathbb{V}[X] + b^2 \mathbb{V}[Y] + ab \text{Cov}(X,Y)$.

(a) What is the expectation of $X + Y$? How is this quantity distributed?
(b) What is the variance of $X + Y$? How is this quantity distributed?
(c) What is the variance of $-\alpha Y$? How is this quantity distributed?
(d) What is the standard deviation of $\alpha X - \beta Y$? How is this quantity distributed?
(e) What is the joint density $f_{X,Y}(x,y)$? Simplify your solution within reason.

5. Suppose $X$ is distributed with PDF $f_X(x) = \frac{1}{Z} \begin{cases} x : & x \in [0,1) \\ 1 : & x \in [1,3) \\ 0 : & \text{otherwise} \end{cases}$

with some normalization constant $Z$ such that $\int_{\mathbb{R}} f_X(x)dx = \int_{-\infty}^{\infty} f_X(x) = 1$.

(a) What is the value of $Z$? Try to compute this quantity using geometry and using calculus. Make sure your answers agree!
(b) What is the CDF of $X$?
(c) Suppose $Y = \lfloor X \rfloor$, where $\lfloor \cdot \rfloor$ is the *floor* operator that takes any real number onto the largest integer not greater than its argument. What are the PMF and CMF of $Y$?

**Hint**: Recall that for $Y = g(X)$, it holds that $\mathbb{P}(Y = y) = \mathbb{P}(X \in \{x \mid g(x) = y\})$.
(d) Without appealing to computing their values, how do $\mathbb{E}[X]$ and $\mathbb{E}[Y]$ compare? Defend your answer.
(e) **Bonus**: Suppose I draw (sample) a random value $X_1$ with density $f_X$. For any $i > 0$, if $X_i > 1$, I discard it and restart the process for a new $X_{i+1}$ drawn with density $f_X$ with probability $\min(x-1,1)$, otherwise I let $X$ equal $X_i$.

Prove that $f_X(x) = f_X(x) \mathbb{P}(X_1 \text{ is discarded}) + f_X(x) \mathbb{P}(X_1 \text{ is not discarded} | X_1)$.
(f) **Bonus**: Use the solution to part (e) to derive $f_X(\cdot)$.

Please consult previous recitations and textbook problems for additional practice.