# Recitation 8: Monte Carlo Simulation*

Brown University CS145: Probability & Computing

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## 1 Overview of Monte Carlo Framework:

Why do we use simulation?

- To understand complex stochastic systems
- To control complex stochastic systems

Such systems are often too complex to be understood or controlled using analytic or numerical methods

- Analytical Methods
  - can examine many decision points at once
  - but limited to simple models
- Numerical Methods
  - can handle more complex models but still limited
  - often have to repeat computation for each decision point
- Simulation
  - can handle very complex and realistic systems
  - but has to be repeated for each decision point

*Material adapted from IEOR E4703, Columbia University, Fall 2004.

# 2 Examples

## 2.1 Monte Carlo Integration

Suppose we want to compute

$$\theta = \int_0^1 g(x)dx$$

If we cannot compute $\theta$ analytically, then we could use numerical methods. However, we can also use simulation and this can be especially useful for high-dimensional integrals. The key observation is to note that

$$\theta = E[g(U)]$$

where $U \sim U(0,1)$. We can use this observation as follows

1. Generate $U_1, U_2, \cdots, U_n \sim i.i.d \quad U(0,1)$
2. Estimate $\theta$ with

$$\hat{\theta}_n = \frac{g(U_1) + \cdots + g(U_n)}{n}$$

There are two reasons why $\hat{\theta}_n$ is a good estimator

1. $\hat{\theta}_n$ is **unbiased**, $E[\hat{\theta}_n] = \theta$
2. $\hat{\theta}_n$ is **consistent**, i.e $\hat{\theta}_n \to \theta$ as $n \to \infty$ with probability 1.

In probability theory, the **law of large numbers (LLN)** is a theorem that describes the result of performing the same experiment a large number of times. According to the law, the average of the results obtained from a large number of trials should be close to the expected value, and will tend to become closer as more trials are performed.

So suppose we want to estimate $\theta = \int_1^3 (x^2 + x)dx$, we can write

$$\theta = 2\int_1^3 \frac{x^2 + x}{2}dx = 2E[X^2 + X]$$

Where $X \sim U(1,3)$. See *lln.m*

## 2.2 Monte Carlo Integration in 2-D

Suppose now that we wish to approximate

$$\theta = \int_0^1 \int_0^1 g(x_1, x_2)dx_1 dx_2$$

Then we can write $\theta = E[g(U_1, U_2)]$ where $U_1, U_2$ are $i.i.d \quad U(0,1)$ random variables. As before we can estimate $\theta$ using simulation by performing the following steps

1. Generate $2n$ independent $U(0,1)$ variables
2. Compute $g(U_1^i, U_2^i)$ for $i = 1, \ldots, n$

3. Estimate $\theta$ with

$$\hat{\theta}_n = \frac{g(U_1^1, U_2^1) + \cdots + g(U_1^n, U_2^n)}{n}$$

Suppose we want to estimate

$$\theta = \int_0^1 \int_0^1 (4x^2y + y^2)dxdy$$

The true value of $\theta$ is 1, see *lln.m*

## 2.3 Monte Carlo Integration in general

If we want to estimate

$$\theta = \int \int_A g(x,y)f(x,y)dxdy$$

where $f(x,y)$ is a density function on $A$, then we observe that

$$\theta = E[g(X,Y)]$$

where $X, Y$ have joint density $f(x,y)$. To estimate $\theta$ using simulation, we

1. Generate $n$ random vectors $(X, Y)$ with joint density $f(x,y)$
2. Estimate $\theta$ with

$$\theta_n = \frac{g(X_1, Y_1) + \cdots + g(X_n, Y_n)}{n}$$

Normally $n$ should be a very large number, otherwise we cannot apply LLN. An example is illustrated in *lln.m*, where we vary the sample size $n$ in section 2.2 and plot the estimation in figure 1, showing that when $n$ gets large, the estimation converges and when $n$ is small, the estimation will be noisy.

## 2.4 Inventory Problem

A retailer sells a perishable commodity and each day he places an order for $Q$ units. Each unit that is sold gives a profit of 60 cents and units not sold at the end of the day are discarded at a loss of 40 cents per unit. The demand, $D$, on any given day is uniformly distributed on $[80, 140]$. How many units should the retailer order to maximize expected profit?

**solve analytically** Let $P$ denote profit, then

$$P = \begin{cases} 0.6Q & D \geq Q \\ 0.6D - 0.4(Q - D) & D \leq Q \end{cases} \tag{1}$$

We can write the expected profit as

$$E[P] = \int_Q^{140} \frac{0.6Q}{60}dx + \int_{80}^{Q} \frac{0.6x - 0.4(Q-x)}{60}dx$$

We use calculus to find the optimal $Q^* = 116$

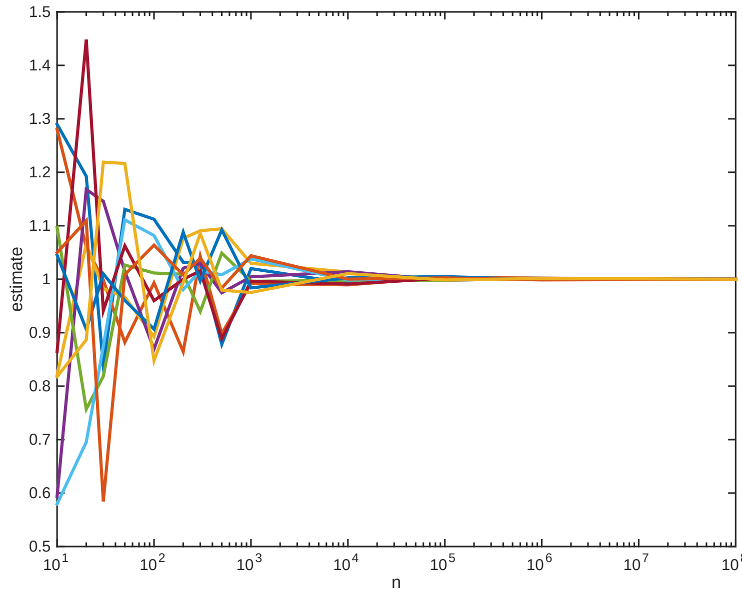


Figure 1: sample size VS estimation

**use simulation**

1. set $Q = 80$
2. Generate $n$ replications of $D$
3. For each replication, compute profit or loss $P_i$
4. Estimate $E[P]$ with $\frac{\sum P_i}{n}$
5. Repeat for different values of $Q$
6. Select the value that gives the biggest estimated profit.

See *inventory.m*