Recitation 6: Derived Distributions

Brown University CS145: Probability & Computing

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Question 1:

Suppose that a stick is randomly broken in two places. What is the probability that the three pieces form a triangle?

Solution

Refer to http://www.math.uah.edu/stat/buffon/Triangles.html for discussion and simulations.

The three pieces form a triangle if and only if the triangle inequalities hold: the sum of the lengths of any two pieces must be greater than the length of the third piece. With a total length of one, this is equivalent to each piece having length strictly less than 1/2.

If $x < y$, the three pieces have lengths x, $y - x$, and $1 - y$. So a triangle can be formed if $x < 1/2$, $y - x < 1/2$, and $y > 1/2$.

Symmetrically, a triangle can be formed if $y < 1/2$, $x - y < 1/2$, and $x > 1/2$.

Sketching those regions (as on the above website) shows that the probability of forming a triangle is 1/4.

Question 2:

(4.1 from Bertsekas and Tsitsiklis) If X is a random variable that is uniformly distributed between -1 and 1, find the PDF of $\sqrt{|X|}$.

Solution

See the solution for 4.1, available at http://athenasc.com/prob-solved_2ndedition.pdf.

Question 3:

Let $X_1, \ldots, X_n$ be independent random variables, and suppose X_i has exponential distribution with rate $\lambda_i, i = 1, ..., n$. Find the distribution of X_{min}, the minimum of $X_1, ...X_n$.

Solutions

For $i = 1, ..., n$, the c.d.f of X_i is

$$F_i(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-\lambda_i x} & x \geq 0 \end{cases}$$

Since X's are non-negative, so is their minimum. So X_{min} has c.d.f

$$F_{min}(x) = 0, \quad (x < 0)$$

For $x \geq 0$

$$\begin{aligned} F_{min}(x) &= p(x_{min} <= x) \\ &= 1 - p(x_{min} > x) \\ &= 1 - (p(x_1 > x)p(x_2 > x) \cdots p(x_n > x) \\ &= 1 - (1 - F_1(x))(1 - F_2(x)) \cdots (1 - F_n(x)) \\ &= 1 - e^{-\lambda_1 x}e^{-\lambda_2 x} \cdots e^{-\lambda_n x} \\ &= 1 - e^{-(\lambda_1 + ... + \lambda_n)x} \end{aligned}$$

This is the c.d.f. of the exponential distribution with rate $\lambda_1 + ... + \lambda_n$. So the minimum of independent exponential variables is simply a new exponential variable with rate the sum of the rates λ_i.