Midterm Take-Home

Due: Tuesday, October 26 @ 11:59pm
Remember to show your work for each problem to receive full credit.

Exam instructions (please read carefully):

• **Exam structure.** This exam contains 5 problems. Choose any 4 of the 5 problems. Each problem is worth 25 points. *If you submit more than 4 questions, we will grade only the first 4 of the answers.*

• **Allowed resources.** You are allowed to consult the course web site, your notes, and the course textbook. You are not allowed to use other resources, such as other books, other web sites, or other people.

• **Non-collaboration.** No collaboration is permitted on this exam. It is trusted that you will not discuss this exam or related course material with any other person (classmate or otherwise). You must abide by the Brown University Academic Code concerning examinations, quizzes, and tests (see [http://goo.gl/mQtfSa](http://goo.gl/mQtfSa)).

• **Questions.** There will be no TA hours during this week. For any clarification of possible textual ambiguities email cs1450headtas@lists.brown.edu with your question. *We will post clarifications (anonymously) to Campuswire.*

• **Handing in.** Submit to Gradescope by the due date.

PLEASE SIGN (failure to sign voids the exam): I solemnly state that I have abidden by the exam instructions stated above, including the tenets of the Brown University Academic Code concerning examinations, quizzes and tests.
Problem 1 (25 points)

The amount of traffic in winter in Providence depends on whether it snows or not, and it snows with probability \( \frac{1}{3} \). With no snow, the time for a taxi to drive from Brown to the airport is exponentially distributed with a mean of 15 minutes. With snow, the travel time is exponentially distributed with a mean of 25 minutes. Given that it takes you 23 minutes to reach the airport, what is the probability that it is snowing?
Problem 2 (25 points)

Two points $X$ and $Y$ are chosen independently and uniformly at random from the interval $[0, 1]$. Let $D$ be the random variable which is equal to the distance of the two points.

1. What is the expected distance between the points ($E[D]$)?

   *Hint: use the conditional probability distribution, $P_{X|Y=y}$.*

2. Use Markov’s inequality to find an upper bound on $P[D \geq 2/3]$. 
Problem 3 (25 points)

A fair coin is tossed until heads appears 40 times. Let $X$ be the number of tosses required.

(a) Find the distribution of $X$, its expected value $\mathbb{E}[X]$, and variance $\mathbb{V}[X]$. Use Chebyshev’s inequality to find an upper bound on $\mathbb{P}(X \leq 70)$.

(b) What continuous distribution can be used to approximate the distribution of $X$? State the values of any parameters necessary to specify this distribution.

(c) Now find the expectation and variance of $X$ given that “in the first 10 tosses the coin never lands heads.” Again, Use Chebyshev’s inequality to find an upper bound on

$$\mathbb{P}(X \leq 70 \mid \text{the coin never lands heads in the first 10 tosses})$$
Problem 4 (25 points)

Alice is sending Bob ‘yes’/‘no’ messages, encoded as 1 and $-1$, respectively.

However, the telephone line Alice is using adds noise to the value of her bit. The noise is normally distributed with mean $\mu = 0$ and variance $\sigma^2 = 4$. To account for the noise, the receiver returns 1 if the message combined with the noise is positive, and $-1$ if the total received signal is negative.

(a) If Alice sends a single bit, what is the probability that Bob receives it incorrectly?

(b) Bob wants to improve the accuracy of the communication protocol. He wants Alice to upgrade her telephone to a model which will be received with noise with variance $\sigma^2 = 2$. What is the probability of receiving an incorrect message with this telephone?

(c) Alice has a different idea to improve Bob’s reception. She wants to transmit each message three times in a row (using her original telephone), and Bob will receive each as a 1 or $-1$. Bob will then take the message to be the median of the three messages received. What is the probability of receiving an incorrect message using this method?
Problem 5 (25 points)

Given two discrete random variables, $X$ and $Y$, we can define the conditional expectation of $X$ given $Y = y$ as follows:

$$E_X[X|Y = y] = \sum_x x P(X = x|Y = y)$$

(a) Show that $E[X] = E_Y[E_X[X|Y]]$.

Now choose $X$ uniformly at random from the set $\{1, 2, \ldots, 6\}$. Then choose $Y$ uniformly at random from $\{1, 2, \ldots, X\}$.

(b) Find the joint pmf, $p_{X,Y}(x, y)$, of $X$ and $Y$.

(c) Find the conditional pmf, $p_{X|Y}(x|y)$, of $X|Y$.

(d) Find $E[Y|X]$ and use that to find $E[Y]$. 