Problem 1

Recall that we denote the size, or \textit{cardinality}, of a set \( X \) as \(|X|\). So, by definition,

\(|\emptyset| = 0, \quad |\{1, 2, 3\}| = 3, \quad \text{etc.}\)

Consider a class of students who each took an Algebra and a Biology exam.

(a) Let \( S \) be the set of all students; among them let \( A \subseteq S \) be the subset of students who failed Algebra, and \( B \subseteq S \) the subset of them who failed Biology. Draw a Venn diagram of these sets. Is the following statement always true? If not, can you correct it such that it is always true?

\(|A \cup B| = |A| + |B|\).

(b) Assume that we have 50 students (\(|S| = 50\)), and that 3 have failed Algebra, 5 have failed Biology and 1 student has failed both. How many people have failed either Algebra or Biology? If we take a student uniformly at random from this class, what is the probability that they have failed at least one of the exams?

(c) A TA needs to group the students who failed either Algebra or Biology each with a student who passed both into pairs of `study buddies.’ Each pair should have one student who failed either Algebra or Biology and one student who passed both, and no two different pairs of students should share the same students. Using the setup from part (b), how many unique ways are there to pair the students up?

(d) For two events \( A \) and \( B \) in a probability space, prove that

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B). \]

Hint: The part of \( A \) that is not contained in \( B \) can be written as \( A \setminus B \).

(e) In this subproblem, we will walk through a proof by \textbf{induction}\n
i. For an event \( A_1 \), we know that \( P(A_1) \geq P(A_1) \). Show that for events \( A_1, A_2, \)
\[ P(A_1) + P(A_2) \geq P(A_1 \cup A_2). \]

ii. Assume \( \sum_{i=1}^n P(A_i) \geq P(\bigcup_{i=1}^n A_i) \). Show that for \( n + 1, \sum_{i=1}^{n+1} P(A_i) \geq P(\bigcup_{i=1}^{n+1} A_i) \)

iii. For any \( n \) events \( A_1, A_2, \ldots, A_n \) in a probability space, conclude that

\[ P(A_1) + P(A_2) + \cdots + P(A_n) \geq P(A_1 \cup A_2 \cup \cdots \cup A_n). \]
Problem 2

You have a standard deck of 52 playing cards, shuffled in a random order, and you draw cards one at a time from the deck.

(a) What is the probability of drawing five cards in consecutive increasing order (i.e. the value of the second card is one greater than the value of the first and so on), where jacks, queens, kings, and aces have values 11, 12, 13, and 14, respectively.

(b) What is the probability of drawing a hand that can be arranged to be in consecutive increasing order?

(c) Without looking at the actual numbers obtained above, which probability is higher between (a) and (b)? Can you provide an intuitive explanation as to why?

(d) Now, assume that after each card is drawn, you shuffle it back into the deck. Now what is the probability that the five cards you drew were in increasing consecutive order?

(e) Without looking at the actual numbers obtained above, which probability is higher between (a) and (d)? Can you provide an intuitive explanation on why?

(f) Given that you drew five cards in increasing consecutive order, what is the probability that all of the cards are red or you have exactly three diamond suits (n.b., the diamond suit is red)?
Problem 3

Travis has taken up a job as a taxi driver in big, boisterous Square City, but he’s not sure which route to take!

The area of the city that his taxi serves is organized as a $10 \times 10$ block grid of streets and avenues. Travis initially starts at the South-West corner of the $10 \times 10$ grid $(0, 0)$, and needs to drop off a passenger at the North-East corner of the city $(10, 10)$. He wants to take the shortest possible path, but realizes there is more than one shortest path! Keep in mind that each road is the same length.

(a) How many unique shortest paths could Travis take to reach his final destination?

(b) Now Travis has two passengers. He starts at the same location, but needs to drop off the first passenger at $(5, 5)$, and the second at $(10, 10)$. If he’s traveling the shortest possible distance, how many unique paths could he drop off these passengers? (He has to drop off the passengers in order)

(c) It turns out that today is the great Festival of Triangles in Square City! Hence, there is a temporary wall running straight from $(0, 0)$ to $(10, 10)$. Taxis are not allowed to cross this wall, but they can visit the intersections the wall runs through. On this festive day, if Travis starts at $(0,0)$, how many different paths could he take to drop off his passenger at $(10, 10)$ without crossing the temporary wall? Is this half of the answer you calculated in part (a), and why or why not?
Problem 4

A stack (LIFO) is a type of data structure where elements can only be added to or removed from the top of the stack. An example is adding or removing plates from a stack of plates. You can refer to https://users.ece.cmu.edu/~koopman/stack_computers/sec1_2.html for more information about stacks, but for this problem, we just consider adding to or removing from the stack, as well as the current arrangement of elements in the stack.

At various points throughout the week, the $N$ students enrolled in CS1450 hand in their homework assignments to Gradescope. The $N$ assignments accumulate in a stack (Last-In-First-Out), and are numbered according to order of arrival by the natural numbers, i.e. 1st to arrive is labeled 1, 2nd to arrive is labeled 2, etc. Throughout the week, whenever the TAs have time, they will remove the most recently submitted ungraded assignment and grade it (the assignment on the top of the stack is removed and then graded).

Mid-week, the TAs announce that the $k$th submitted assignment has been graded, but neglect to provide further information. After the announcement is made, the students wonder which of the $N$ assignments are still ungraded, and what are the possible orders for grading the remainder. Given this information, how many such after-announcement grading orders are possible, if:

(a) $k = N$?

(b) $k = N - 1$?

(c) $k = 1$, and the TAs always grade an assignment immediately after it is submitted?
Problem 5

Darius Dice conducts an experiment in which he rolls a fair six-sided die. If he rolls any number other than a 6, he records the result and rolls the die again. If he rolls a 6, he records the result and stops. If Darius rolls the die 10 times in a row without rolling a 6, he becomes bored and stops anyway. To show your work below, submit your code as well as any derivations.

(a) What is the probability that Darius rolls at least one 6? Derive an equation for this probability, and also compute its numeric value.

(b) Write a program to determine the probability that Darius rolls at least one 5 by counting the number of outcomes where this occurs. Hint: You can verify your code by hand-checking the result for a smaller maximum number of die rolls.

(c) Write a program to determine the probability that Darius rolls strictly more 4s than 3s.

(d) Ten die rolls can sum to at most 60. Let R denote the sum of the ten rolls. What is the range of possible sums of rolls (the sample space!) that Darius can achieve? Compute and plot the probabilities that the sum of Darius’s rolled dice equal each integer in that range.

(e) Cyrus rolls dice similarly, but stops if he rolls a 1 (or after 10 rolls). What is the range of possible sums that Cyrus can roll? Compute and plot the probability that the sum of Cyrus’s dice equals each integer in that range.

(f) Cyrus challenges Darius to a game in which they each roll dice as described above, and the winner is the person who rolls the sum closest to some target integer x, without going over. If they roll the same sum, or both go over, no one wins. Compute and plot the probability that Darius wins, and the probability that Cyrus wins, for all targets 1, 2, \ldots, 60. Attach your plot below.