Conditional Probabilities

Definition

The conditional probability that event $E$ occurs given that event $F$ occurs is

$$\Pr(E \mid F) = \frac{\Pr(E \cap F)}{\Pr(F)}.$$  

The conditional probability is only well-defined if $\Pr(F) > 0$.

- By conditioning on $F$ we restrict the sample space to the set $F$.
- $\Pr(E \mid F)$ defines a proper probability function on the sample space $F$. 
Bayes’ Law

Theorem (Bayes’ Law)

Assume that $E_1, E_2, \ldots, E_n$ are mutually disjoint sets such that $\bigcup_{i=1}^{n} E_i = \Omega$, then

$$
\Pr(E_j \mid B) = \frac{\Pr(E_j \cap B)}{\Pr(B)} = \frac{\Pr(B \mid E_j) \Pr(E_j)}{\sum_{i=1}^{n} \Pr(B \mid E_i) \Pr(E_i)}.
$$
Application: Naive Bayes Classifier

Automatic classifications of objects to categories

- Junk mail filter
- Classify text documents into one of several subjects/topics - politics, business, sport, science, religion,....
- Classify to genres - "editorials", "movie-reviews", "news",....
- Classify opinions: like, hate, neutral,...
- Identify the language of a document: English, French, Chinese,...
- recommendation systems
- ...
Bayes Classifier

• We have a training set of classified documents.
• We assume that the category of a document can be deduced from the words used in the document.
• Training Phase: Learn from the training set the conditional probabilities that a given word appears in a document of a given category - Supervised learning
• Classification Case: Compute the conditional probability that a new document belongs to a given category conditioned on the words that appear (or don’t appear) in the document.
The "bag of words" model

- A document is represented by the set of words in the document.
- We ignore locality relation and number of occurrences of words.
- We clean the documents by removing HTML commands, stop words, "s"'s and "ing"'s, etc. ("tokenizing"), so we are left with the **keywords** of the document.
Bayes Classifier

- \( X_w^d \) - the event ”word \( w \) appears in document \( d \”).

\[
Pr(d \in C_i \mid \bigcap_{w \in d} X_w^d) = \frac{Pr((d \in C_i) \cap (\bigcap_{w \in d} X_w^d))}{Pr(\bigcap_{w \in d} X_w^d)}
\]

\[
= \frac{Pr(\bigcap_{w \in d} X_w^d \mid d \in C_i)Pr(d \in C_i)}{\sum_j Pr(\bigcap_{w \in d} X_w^d \mid d \in C_j)Pr(d \in C_j)}
\]

- The classification of \( d \) is

\[
\arg \max_{C_i \in C} Pr(d \in C_i \mid \bigcap_{w \in d} X_w^d)
\]

- It’s the maximum likelihood category
The "Naive" assumption

- **Problem:** Assume \( n \) categories and a dictionary of \( k \) keywords. Need to estimate \( k = n \cdot 2^k \) probabilities:

\[
Pr(\cap_{w \in d} X_w^d \mid d \in C_i)
\]

- **Practical solution:** Assume that occurrences of words are independent.

\[
Pr(\cap_{w \in d} X_w^d \mid d \in C_i) = \prod_{w \in d} Pr(X_w^d \mid d \in C_i)
\]

\[
Pr(d \in C_i \mid \cap_{w \in d} X_w^d) = \frac{\prod_{w \in d} Pr(X_w^d \mid d \in C_i)Pr(d \in C_i)}{\sum_j \prod_{w \in d} Pr(X_w^d \mid d \in C_j)Pr(d \in C_j)}
\]

- The classification of \( d \) is

\[
\arg \max_{C_i \in C} Pr(d \in C_i \mid \cap_{w \in d} X_w^d)
\]
Using the training data:

- For each category $C \in C$ and keyword $w$ we compute

  $$Pr(\text{page includes } w \mid \text{page in } C) = \frac{|\{d \mid d \in C \cap w \in d\}|}{|\{d \mid d \in C\}|}$$

- For each category $C \in C$, compute

  $$Pr(\text{page in } C) = \frac{|\{d \mid d \in C\}|}{|\{\text{all pages}\}|}$$
Naive Bayes Classifier

- We are looking for the category $C_i^*$ that maximizes

$$Pr(d \in C_i \mid \bigcap_w X_w^d) = \frac{\prod_w Pr(X_w^d \mid d \in C_i)Pr(d \in C_i)}{\sum_j \prod_w Pr(X_w^d \mid d \in C_j)Pr(d \in C_j)},$$

- We only need to consider the nominators. The classification of $d$ is

$$\arg \max_{C_i \in \mathcal{C}} Pr(d \in C_i \mid \bigcap_w X_w^d) = \arg \max_{C_i \in \mathcal{C}} \prod_{w} Pr(X_w^d \mid d \in C_i)Pr(d \in C_i)$$
Two technical problems

- **Underflow prevention:** Multiplying lots of probabilities can result in very small quantities and floating-point underflow.

- **Solution:** We do all computations in log’s. The classification of \(d\) is

\[
\arg \max_{C_i \in \mathcal{C}} \Pr(d \in C_i \mid \cap_w X^d_w) = \arg \max_{C_i \in \mathcal{C}} \left[ \log \Pr(d \in C_i) + \sum_w \log \Pr(X^d_w \mid d \in C_i) \right]
\]
• **Zero probability events:** What if \( w \in d \) but we have seen no training documents in category \( C \) with keyword \( w \)?

• In that case \( Pr(X_w^d \mid d \in C_i) = 0 \), cannot take the log

• Zero probabilities cannot be conditioned away, no matter the other evidence!

• **Solution:** Smoothing

\[
Pr(X_w^d \mid C) = \frac{|\{d \mid (d \in C) \cap (w \in d)\}| + 1}{|\{d \mid d \in C\}| + n}
\]

• No zero probability events.
Naive Bayes Classifier Algorithm

**Training phase:** \( TD \) set of classified documents, \( n \) categories.

- For each category \( C \in C \) and keyword \( w \) compute

\[
Pr(\text{page includes } w \mid \text{page in } C) = \frac{|\{d \mid d \in C \land w \in d\}| + 1}{|\{d \mid d \in C\}| + n}
\]

- For each category \( C \in C \), compute

\[
Pr(\text{page in } C) = \frac{|\{d \mid d \in C\}|}{|TD|}
\]

- Store \( n + nk \) probabilities.

To **classify** a new document \( d \), compute

\[
\arg \max_{C_i \in C} \left[ \log Pr(d \in C_i) + \sum_{w \in d} \log Pr(X^d_w \mid d \in C_i) \right]
\]

Execute \( O(n(|d| + 1)) \) operations