In these notes, we derive Myerson’s payment characterization by way of the envelope theorem. We closely follow the presentation in Quint \cite{1}.

1 Envelope Theorem

In this section, we prove the envelope theorem in its simplest form. In more interesting/complicated versions, the constraint set depends on the parameter $\theta$.

Consider an optimization problem whose objective is parametrized by some $\theta \in \Theta$:

$$V(\theta) = \max_{a \in A} f(a; \theta) \quad (1)$$

Let $A^*(\theta) = \arg \max_{a \in A} f(a, \theta)$, and assume $A^*(\theta)$ is nonempty, with $a^*(\theta)$ be an element of $A^*(\theta)$, so that $V(\theta) = f(a^*(\theta); \theta)$.

**Theorem 1.1.** If $\frac{\partial f(a; \theta)}{\partial \theta}$ exists, for all $a \in A, \theta \in \Theta$, and if $V$ is differentiable at $\theta$, then

$$V'(\theta) = \frac{\partial f(a; \theta)}{\partial \theta} \bigg|_{a = a^*(\theta)} = \frac{\partial f(a^*(\theta); \theta)}{\partial \theta}.$$
Proof. If $V$ is differentiable at $\theta$, then

$$V'(\theta) = \lim_{\epsilon \to 0} \frac{V(\theta + \epsilon) - V(\theta)}{\epsilon} = \lim_{\epsilon \to 0} \frac{V(\theta) - V(\theta - \epsilon)}{\epsilon}.$$  

Case 1: $V'(\theta) \leq \frac{\partial f(a^*(\theta); \theta)}{\partial \theta}$:

Since $V(\theta + \epsilon) = \max_{a \in A} f(a; \theta + \epsilon) \geq f(a^*(\theta); \theta + \epsilon)$, it follows that

$$V'(\theta) = \lim_{\epsilon \to 0} \frac{V(\theta + \epsilon) - V(\theta)}{\epsilon} \geq \lim_{\epsilon \to 0} \frac{f(a^*(\theta); \theta + \epsilon) - f(a^*(\theta); \theta)}{\epsilon} = \frac{\partial f(a^*(\theta); \theta)}{\partial \theta}.$$  

Case 2: $V'(\theta) \geq \frac{\partial f(a^*(\theta); \theta)}{\partial \theta}$:

Similarly, since $V(\theta - \epsilon) = \max_{a \in A} f(a; \theta - \epsilon) \geq f(a^*(\theta); \theta - \epsilon)$, it follows that

$$V'(\theta) = \lim_{\epsilon \to 0} \frac{V(\theta) - V(\theta - \epsilon)}{\epsilon} \leq \lim_{\epsilon \to 0} \frac{f(a^*(\theta); \theta) - f(a^*(\theta); \theta - \epsilon)}{\epsilon} = \frac{\partial f(a^*(\theta); \theta)}{\partial \theta}.$$  

The result now follows, since

$$\frac{\partial f(a^*(\theta); \theta)}{\partial \theta} \leq V'(\theta) \leq \frac{\partial f(a^*(\theta); \theta)}{\partial \theta}.$$

\[\square\]

**Corollary 1.2.** By the fundamental theorem of calculus, for all $-\infty < \theta < \tau < \infty$,

$$V(\tau) - V(\theta) = \int_\theta^\tau \frac{\partial f(a; z)}{\partial \theta} \, dz \bigg|_{a=a^*(z)}$$

**Proof.** By the fundamental theorem of calculus, we obtain

$$V(\tau) - V(\theta) = \int_\theta^\tau V'(z) \, dz \quad \text{(2)}$$

$$= \int_\theta^\tau \frac{\partial f(a; z)}{\partial \theta} \, dz \bigg|_{a=a^*(z)} \quad \text{(3)}$$

\[\square\]

## 2 Myerson’s Payment Characterization

We now set out to prove Myerson’s payment characterization theorem using the envelope theorem. Specifically, we will prove that the DSIC assumption implies the payment formula, via the envelope theorem.

**Proof.** Fix $v_{-i} \in T_{-i}$. Consider bidder $i$ with type $v_i$.

Bidder $i$ seeks to optimize the following function:

$$\max_{t_i \in T_i} v_i x_i(t_i) - p_i(t_i).$$

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We call this function bidder $i$’s value function, and denote it by $V_i(v_i)$. By the DSIC assumption, bidder $i$’s value function is maximized at $v_i$: i.e., $V_i(v_i) = v_i x_i(v_i) - p_i(v_i)$.

Further, by the envelope theorem,

$$V_i'(v_i) = \frac{\partial}{\partial v_i} \max_{t_i \in T_i} v_i x_i(t_i) - p_i(t_i)$$

$$= \frac{\partial}{\partial v_i} v_i x_i(t_i) - p_i(t_i) \bigg|_{t_i=v_i}$$

$$= x_i(t_i) \bigg|_{t_i=v_i}$$

$$= x_i(v_i).$$

More specifically, $V_i'(v_i, v_{-i}) = x_i(v_i, v_{-i})$.

But then, by the fundamental theorem of calculus,

$$V_i(v_i, v_{-i}) - V_i(v_{-i}, v_{-i}) = \int_{v_{-i}}^{v_i} V_i'(z, v_{-i}) \, dz$$

$$= \int_{v_{-i}}^{v_i} x_i(v_i, v_{-i}) \, dz.$$  \hspace{1cm} (4)

(5)

Finally, since $V_i(v_i, v_{-i}) = v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i})$, it follows that

$$v_i x_i(v_i, v_{-i}) - p_i(v_i, v_{-i}) - \left( v_{-i} x_i(v_i, v_{-i}) - p_i(v_{-i}, v_{-i}) \right) = \int_{v_{-i}}^{v_i} x_i(v_i, v_{-i}) \, dz.$$  \hspace{1cm} (6)

In other words,

$$p_i(v_i, v_{-i}) = v_i x_i(v_i, v_{-i}) - \int_{v_{-i}}^{v_i} x_i(v_i, v_{-i}) \, dz + p_i(v_{-i}, v_{-i}) - v_{-i} x_i(v_{-i}, v_{-i}).$$  \hspace{1cm} (7)

\[\square\]

References