Game Theory
Goals

- Define “game”
- Link games to AI
- Introduce basic terminology of game theory
- Overall: give you a new way to think about some problems
What Is Game Theory?

● Field of work involving games, answering such questions as:

■ How should you play games?

■ How do most people play games?

■ How can you create a game that has certain desirable properties?
What Is a Game?
What Is a Game?

It is a situation in which there are:

- **Players**: decision-making agents

- **States**: where are we in the game?

- **Actions** that players can take that determine (possibly randomly) the next state

- **Outcomes** or **Terminal States**

- **Goals** for each player (give a score to each outcome)
Example: Rock-Paper-Scissors

- **Players?**
  - 2 players
- **States?**
  - before decisions are made, all possibilities after decisions are revealed
- **Actions?**
  - \{Rock, Paper, Scissors\}
- **Outcomes?**
  - \{(Rock, Rock), (Rock, Paper), \ldots, (Scissors, Scissors)\}
- **Goals?**
  - Maximize score, where score is 1 for win, 0 for loss, ½ for tie
Example: Classes

- **Players?**
  - All students, instructor(s)

- **States?**
  - Points in time

- **Actions?**
  - Students: `study(time), doHomework(), sleep(time)`
  - Instructors: `chooseInstructionSpeed(speed), review(topic, time), giveExample(topic, time)`

- **Outcomes?**
  - Amount learned by students, grades, time spent, memories made

- **Goals?**
  - Attain some ideal balance over attributes that define the outcomes
Why Study Game Theory in an AI Course?

- making good decisions ⊆ AI
- making good decisions in games ⊆ Game Theory
- AI often created for situations that can be thought of as games
How Do Games Differ?
## Sequential vs. Simultaneous Turns

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Sequential vs. Simultaneous Turns

Sequential

Simultaneous

- Chess piece
- Graduation cap
- People holding puzzle pieces
- People holding apples and wheat

- Money in an envelope
Constant-Sum vs. Variable-Sum

Constant-Sum

Variable-Sum
Constant-Sum vs. Variable-Sum

Constant-Sum

Variable-Sum
Restricting the Discussion

- 2-player, one-turn, simultaneous-move games
"Normal Form" Representation

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Strategies

- **Strategy** = A specification of what to do in every single non-terminal state of the game

- Functions from states to (probability distributions over) legal actions
  - Pure vs. Mixed

Examples:
- Trading: I’ll accept an offer of $20 or higher, but not lower
- Chess: Full lookup table of moves and actions to make
What’s the best strategy in rock-paper-scissors?

- It depends on what the other player is doing!
Best Response

- But if we knew what the other player’s strategy…?
- Then we could choose the best strategy. Now it’s an optimization problem!
Dominated Strategies

- A strategy $s$ is said to be dominated by a strategy $s^*$ if $s^*$ always gives higher payoff.

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A strategy \( s \) is said to be dominated by a strategy \( s^* \) if \( s^* \) always gives higher payoff.
Dominant Strategies

- A strategy is *dominant* if it dominates all other strategies.
## Iterated Dominance

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Iterated Dominance

- *Iterated Elimination of Dominated Strategies (IEDS)*
- Won’t always produce a unique solution
- Common Knowledge of Rationality (CKR)
- “Faithful Approach”
**Conservative Approach: Maximin**

- Ensure the best worst-case scenario possible

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Two Different Approaches

- Faithful approach: assume CKR
- Conservative approach: assume nothing, and also avoid risk
Your Turn!

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Nash Equilibrium

- **strategy profile** - specification of strategies for all players

- **Nash equilibrium** - strategy profile such that players are mutually best-responding

- In other words: From a NE, no player can do better by switching strategies alone
### Nash Equilibrium: Stag Hunt

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**Experiment!**
Nash Equilibrium: Stag Hunt

Are there dominated strategies?

Are there more equilibria?

Play B with probability $\frac{1}{3}$, S with probability $\frac{2}{3}$
### Bigger Example of NE

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### How to Find NE

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Properties of NE

- There is always at least one

- If IEDS produces a unique solution, it is a NE.
Next time:

- Learn algorithms for finding maximin pure strategies in sequential, constant-sum, many-turn games