HW9

Due: November 10, 2016

Attach a fully filled-in cover sheet to the front of your printed homework. Your name should not appear anywhere except on the cover sheet; each individual page of the homework should include your Banner ID instead.

While collaboration is encouraged in this class, please remember not to take away notes from any collaboration sessions. Also please list the names and logins of any of your collaborators at the beginning of each homework.

Please monitor Piazza, as we will post clarifications of questions there. You should hand in your solutions by 12:55 to the CSCI 1010 bin on the second floor of the CIT. Late homeworks are not accepted.

Problem 1

The Bega and O’Nion families face a root awakening when a horrible blight strikes their fields, travelling across their alternating fields and killing most of the crops. There are some surviving crops, whose integer weights are given by $S = \{x_1, x_2, \ldots, x_n\}$, where $x_i$ is the weight of the crops on the $i$th lot. The two families are rioting over what is clearly the fault of the other, but Ginger and Rudy are trying to save as much as possible. The only way to carry the crops to a safe silo is an old, beaten-up truck—Rudy’s Dodge Yam. The truck can only carry a weight of $b$; that is, the sum of crop weights for a load must be at most $b$. They want to figure out if they can save everything in $k$ or fewer loads.

The language $\text{TruckLoads}$ is defined as:

$$\text{TruckLoads} = \{ \langle S, b, k \rangle \mid S \text{ can be partitioned into } k \text{ or fewer sets, each of whose weights sums to at most } b \}$$

Prove that $\text{TruckLoads}$ is NP-complete by a reduction from $\text{SubsetSum}$. You can assume without loss of generality that in the original $\text{SubsetSum}$ problem, the target has a value that is at least half of the total sum of the values in the set (it may help to figure out why you can assume this).
Problem 2

The final Myami election is coming up next week, and Walter Chestnutt suspects that farmers’ preferred root vegetable types will be the deciding factor in their votes. As a trained root vegetable geneticist, he knows how to classify the different types: taproot, tuberous root, corm, rhizome, tuber, and bulb. He suspects that this knowledge may be helpful for future gerrymandering initiatives. In the meantime, Walter has done some highly involved analysis of the genetic network graph to come up with a function $f$ that takes in a pair of root vegetables and outputs a number that describes how far away their genomes are from each other. His goal is to partition root vegetables into sets such that the genomic distance between any root vegetables in the same group is less than some threshold he chooses. Formally, the language he wants to decide is:

\[
\text{ClassifyRoots} = \{ (S, f, k, b) \mid S \text{ is a set of root vegetables,} \\
\quad f \text{ is a distance function on } S, \\
\quad k \text{ is an integer,} \\
\quad b \text{ is a real number,} \\
\quad \exists \text{ a } k\text{-partition of } S \text{ into } S_1, S_2, ..., S_k \\
\quad \text{ such that } \forall i, \forall u, v \in S_i, f(u, v) \leq b \}
\]

Crush Walter Chestnutt’s political dreams by showing that ClassifyRoots is NP-complete.

Problem 3

Cass and Ava Yuca recently purchased a farm with a sophisticated irrigation system. Excited to plant potatoes, carrots, corm, and other root vegetables, they want to integrate their plants into the pre-existing irrigation network, known as the dampnet. To do so, they want all plants of each species to be next to every other plant of the same species. Cass and Ava want to determine whether they can figure out how many different species of plants can be planted.

Let $G = (V, E)$ be a graph with vertices $V$ and edges $E$ where each plant is
at a vertex and an irrigation pipe represents each edge. Define the language:

\[ \text{CropIrrigation} = \{ \langle G, k \rangle \mid \exists \text{ a disjoint partition } V_1, V_2, \ldots, V_k \subseteq V \text{ such that } \forall V_i, \forall u, v \in V_i, (u, v) \in E \} \]

That is, CropIrrigation includes \( \langle G, k \rangle \) if the vertices of \( G \) can be partitioned into \( k \) groupings such that any pair of vertices within a grouping is connected by an edge. Prove to Cass and Ava that CropIrrigation is NP-complete.

The following questions are lab problems.

Lab Problem 1

While Ginger and Rudy were off gallavanting in the countryside, Ginger 2.0 needed to harvest the latest O’Nion mangelwurzel crop back in Myami. Ginger 2.0 was supposed to pack everything into a two-column rectangular root vegetable crate, but ran into several technical difficulties having to do with onionialized variables\(^1\). Root vegetable crates contain several linings that absorb moisture and are thus crucial for ensuring freshness. Unfortunately, while Ginger 2.0 was fixing variables, busy beavers chewed through all of the linings and stole some of the mangelwurzel.

Each lining now has some holes that align with a two-column, \( N \)-row grid. The heads of the O’Nion family are coming to inspect the root vegetable crate in the afternoon. Luckily, they have notoriously poor eyesight and will only be able to tell that something is wrong if they can see all the way through the linings to the bottom of the crate. Ginger 2.0 needs to stack the linings such that for every position in the grid, at least one lining covers the position so that the O’Nions cannot see the bottom of the crate through the linings. Ginger 2.0 is allowed to flip a lining along its vertical axis, but cannot rotate it or manipulate it in any other way, because otherwise the linings wouldn’t fit in the crate. See Figure 1 for an example of a valid way to manipulate the lining.

We define AcceptCrate to be the language consisting of sets of crate linings so that an arrangement exists where every possible hole is covered. Prove that AcceptCrate is NP-complete.

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\(^1\)This is an error in languages such as OYaml, Rooty on Rails, and LotusScript.\(^2\)

\(^2\)This is an actual programming language.
**Hint:** Think of the orientation of each container as an assignment to a set of Boolean variables.

![Figure 1: A root vegetable crate lining’s two possible configurations.](image)

**Lab Problem 2**

Rudy often sends Ginger letters with romantic messages. Knowing that her overprotective parents read her letters to make sure they only concern the latest harvest, Rudy has developed a way to send messages that Ginger’s parents cannot recognize. Conveniently for him, the O’Nion parents are all colorblind. Rudy prints the romantic message in a different color ink, so that Ginger can read the message without her parents being able to. He surrounds these differently-colored characters with harsh criticisms of the O’Nions’ farming techniques to convince Ginger’s parents that the only feelings involved are hatred and rivalry.

One fateful day, Ginger comes across such a letter while she’s out in the Myami marketplace. She picks it up and reads it. At first glance, it appears to be a memo concerning Walter Chestnutt’s strategies in the upcoming election. However, she realizes that it’s from Rudy, and that it has an encoded message in which Rudy professes his love for Walter Chestnutt! Ginger is devastated and runs home, finally realizing that her parents were right about the nefarious Begas all along. The O’Nions are surprised that Ginger is upset by such an inane political letter, but she doesn’t tell them
about the secret message. They’re determined to look for hidden meaning, and they examine all of Ginger’s other letters from Rudy to see if they contain a hidden message surrounded by other characters. They decide to look for a message that is contained in all of the letters. Prove to them that the task of finding a common message is NP-complete.

Define the language:

\[ \text{LOVELETTER} = \{ \langle \Sigma, R, k \rangle \mid \Sigma \text{ is a finite alphabet and} \]
\[ R \text{ is a set of strings over } \Sigma \]
\[ \text{such that } \exists w \in \Sigma^* \text{ such that } |w| \geq k \]
\[ \text{and } \forall x \in R, w \text{ is a subsequence of } x \} \]

A string \( w = w_1w_2\ldots w_n \) is a subsequence of \( x = x_1x_2\ldots x_m \) if for all \( 1 \leq i \leq n, w_i = x_{j_i} \) where \( j_i < j_{i+1} \). In other words, a string \( w \) is a subsequence of \( x \) if all characters in \( w \) appear in \( x \) in the same order as they are in \( w \). For example, if \( w = aaab \), then \( w \) is a subsequence of \( acdaccab \). A subsequence is not the same as a substring. A substring is valid if all of its characters remain adjacent; a subsequence, on the other hand, is valid if its characters occur in order, possibly with other characters in between them.

**Hint:** Construct a reduction from \textsc{VertexCover}. Let the alphabet used for \textsc{LoveLetter} correspond to the vertices in the graph for \textsc{VertexCover}. The subsequence should consist of all vertices not included in the vertex cover.

**Lab Problem 3**

Define the **kernel** of a directed graph \( G = (V, E) \) to be some \( K \subseteq V \) such that:

1. \( \forall k_1, k_2 \in K, (k_1, k_2) \notin E \), and
2. \( \forall v \in V - K, \exists k \in K \text{ such that } (k, v) \in E. \)

Let \( L \) be the language of all directed graphs \( G \) which have a kernel. Show that \( L \) is NP-complete by a reduction from 3SAT.

**Hint:** You will need a node for every variable and its complement along with three nodes for every clause. The kernel, if it exists, should contain at least the satisfying assignment of the Boolean formula. It may be helpful to review the proof of NP-completeness for 3-colorability.