Problem 1

The O’Nion family establishes a private language for communicating their top secret mangelwurzel harvesting strategies. They need to guarantee that this need-to-know information is only accessible to immediate members of the O’Nion clan. Ginger O’Nion is tasked with building an interpreter to determine the veracity or falsehood of a purported O’Nion communiqué.

All O’Nion messages are in the language $L$ of the regular expression:

$$(11)^*(01 \cup 001)^*$$

Any message outside of this language (that is, a message in the language $L^C$) is assumed to be the work of covert agents or ne’er-do-well Begas.

a. Provide an NFA whose language is $L$.

b. Provide a regular expression for the language $L^C$. (Show your work.)
Problem 2

Let $B$ be a language, and $A$ and $C$ be regular languages over $\Sigma = \{0, 1\}$, with $A \subset B \subset C$. Which of the following statements is true? Prove your answer.

(i) $B$ must be regular.
(ii) $B$ is regular in some cases, and not regular in some cases.
(iii) $B$ must not be regular.

Problem 3

For each of the following languages, either find a regular expression for the language, or prove that it is not regular. The alphabet is $\{0, 1\}$ unless otherwise specified.

a. $L_a = \{w \mid w$ has an even total number of $1$s\}

b. $L_b = \{xx \mid x$ is any binary string\}

c. $L_c = \{x = y+z \mid x, y, z$ are binary integers, and $x$ is the sum of $y$ and $z\}$. The alphabet is $\{0, 1, =, +\}$.

The following questions are lab problems.

Lab Problem 1

The competition between Ginger and Rudolph has grown too destructive. In their maddening quest to become the biggest farmer in Myami, they have been poisoning one another’s crops with those disgusting non-root vegetables. Food supplies are dwindling, and the mayor of Myami decides to step in. She mandates that they alternate their crops on the same row of farm-land so that they can’t poison the other’s crops without hurting their own. Ginger gets the odd spaces and Rudolph the even ones. To further quell competition, she requires that Ginger and Rudolph grow exactly the same number of crops.

Begrudgingly, they work together to plan their collective farm. They need to leave some empty space for the crops to grow, so they represent their
farmland as a binary string, with 1s representing crops and 0s representing empty space. Farmland is only acceptable if Ginger and Rudolph each grow the same number of crops, where Ginger’s crops are in the odd indices and Rudolph’s are in the even ones.

More formally, the language of acceptable farmland, \( L \), is defined as:
\[
L = \{ w \mid w \text{ contains the same number of 1s in even and odd indices} \}
\]

Use the pumping lemma to show that the language of acceptable farmland is not regular.

**Lab Problem 2**

Recall that a *pumping length* for a language \( A \) is a positive integer \( p \) such that all strings \( s \in A \) with \( |s| \geq p \) can be written in the form \( xyz \), where

(i) \( |xy| \leq p \),

(ii) \( |y| \geq 1 \),

(iii) and \( xy^iz \in A \) for all \( i \geq 0 \).

Also, recall from class that if \( A \) is finite with its longest string of length \( \ell \), \( p = \ell + 1 \) is a valid pumping length for \( A \), because there are no strings \( s \in A \) with \( |s| \geq \ell + 1 \), which makes it vacuously true that all such strings satisfy the three conditions above.

The pumping lemma states that every regular language has a pumping length.

The *minimum pumping length* of a language \( A \), \( p_{\min} \), is the smallest pumping length for \( A \). Note that this implies every integer \( p \geq p_{\min} \) is also a valid pumping length for \( A \).

For example, if \( A = ab^* \) the minimum pumping length is two. To justify this, note that the string \( s = a \) is in \( A \) yet cannot be pumped at all; writing it as \( xyz \) we must have \( x = \epsilon, y = a, z = \epsilon \), and then \( xz \) is not in \( A \). So \( 1 \) is not a pumping length. But \( 2 \) is a pumping length, because for any string \( |s| \geq 2 \) we can take \( x = a, y = b, \) and \( z \) to be everything else, and we have that \( |xy| \leq 2, |y| \geq 1, \) and \( xy^iz \in A \) for all \( i \geq 0 \).

For each of the following languages, give the minimum pumping length \( p_{\min} \) and prove your answer.
Lab Problem 3

Not to be outmaneuvered by the O’Nions, the Begas decide to invent their own secret language. Now that they must share land, it is especially imperatived to guard sensitive stratagems. The Bega language is so secret, in fact, that no DFA can recognize it. Rumor has it that Rudolph has been developing a new technology that can recognize nonregular languages... He has come up with a potential language \( F \) and has enlisted you to help him prove that none of the O’Nion “hackriculturists” can recognize \( F \) with a DFA.

The language \( F \) is defined over the alphabet \( \Sigma = \{a, b, c\} \) as:

\[
F = \{a^i v \mid i \geq 0, v \text{ is a sequence of } b \text{ and } c \text{, and if } i = 1 \text{ then } v \text{ is of the form } xx: \text{ it consists of two identical binary strings}\}.
\]

a. Show that \( F \) is not regular.

b. Give a pumping length \( p \) and demonstrate that, for all strings \( w \in F \) such that \( |w| \geq p \), we can write \( w = xyz \) with \( |xy| \leq p \) and \( |y| \geq 1 \) such that \( xy^iz \in F \) for all \( i \geq 0 \). In other words, you can see that the pumping lemma is not helpful for proving \( F \) is not regular.

c. Explain why parts (a) and (b) do not contradict the pumping lemma.