CSCI 1010
Models of Computation

Lecture 21
Approximation Algorithms
Overview

• VERTEX COVER optimization problem
• 2-approximation to VERTEX COVER
• HAMILTONIAN PATH
• Traveling Salesperson optimization problem (TS)
• 2-approximation to Euclidean TSP
• No P-time k-approximation algorithm, k fixed, exists for general TSP unless P=NP.
VERTEX COVER Decision Problem

VERTEX COVER (VC)

Instance: \(<G,k>\) where \(G = (V,E)\) and \(k\) an integer.

Answer: “Yes” if there is an \(S \subseteq V, |S| = k\), such that every edge is incident on a vertex in \(S\).

Theorem VC is \(\text{NP}-\text{complete}\).
VERTEX COVER Optimization

• Let $G = (V,E)$ and $VC_{opt} : \{G\} \rightarrow \mathbb{N}$, $\mathbb{N} = \{0,1,2, \ldots \}$ where $VC_{opt}(G) = $ size of minimal cover for G.

• $VC_{opt}$ is $\text{NP}$-hard because if $VC_{opt}$ is in $\text{P}$-time, then $VC$ is in $\text{P}$, as we show:
  – Compute $VC_{opt}(G)$ on $<G,k>$. Accept if $VC_{opt}(G) \leq k$.
  – Thus, if $VC_{opt}$ is in $\text{P}$-time, $\text{P} = \text{NP}$. If $\text{P} \neq \text{NP}$, then $VC_{opt}$ is not in $\text{P}$-time.
Optimization Problems

• **Minimization (maximization) problem:**
  – find smallest (largest) cost solution to a problem.

• Decision problems abstract optimization problems.

• Approximation to a minimization problem is a solution close to but above the minimum.

• Roles reversed for maximization problem.
P-Time Approximation Algorithms

- For $0 < k < 1$ ($k > 1$) a P-Time k-approximation algorithm for an NP-hard maximization (minimization) problem is an algorithm that runs in polynomial time in the length of the input and produces a solution whose cost $C$ satisfies

$$C^* \geq C \geq kC^* \quad (C^* \leq C \leq kC^*)$$

where $C^*$ is the cost of the optimal solution.
Approximating SATISFIABILITY

SATISFIABILITY

Instances: Literals $X = \{x_1, \bar{x}_1, x_2, \bar{x}_2, \ldots, x_n, \bar{x}_n\}$, and clauses $C = (c_1, c_2, \ldots, c_m)$, $c_i$ is a subset of $X$.

Answer: “Yes” if $x_1, x_2, \ldots, x_n$ have assignments such that $\geq 1$ literal in each clause has value 1.

- SATISFIABILITY optimization problem is MAX SAT.
  Goal: Find assignment maximizing the number of satisfied clauses.
  - Can we get to $\frac{1}{2}$ of max in polynomial time?
VERTEX COVER Minimization

VERTEX COVER (VC)

Instance: \(<G, k>\) where \(G = (V, E)\) and \(k\) an integer.

Answer: “Yes” if there is an \(S \subseteq V, |S| = k\), such that every edge is incident on a vertex in \(S\).

• Find a vertex cover that is closest to the smallest vertex cover. Let \(C^*\) be its size.
Approximating VERTEX COVER

Approx-VC(G) ; G = (V,E)

1. C ← φ ; C will be vertex cover
2. E′ ← E
3. while E′ ≠ φ
4. do let (u,v) be arbitrary edge in E′
5. C ← C U {u,v} ; Add its vertices to C
6. Remove edges incident on u or v from E′
7. od
8. Return C
Approximating VERTEX COVER

• Let $C^*$ be size of the minimal VC for $G = (V,E)$.

• Approx-VC($G$) returns a cover $C$. Let $A$ denote edges chosen in step 4. Clearly $|C| = 2|A|$.

• Endpoints of edges in $A$ are disjoint. Since at least $|A|$ vertices are needed to cover them, $|A| \leq |C^*|$.

• Thus, $|C| \leq 2|C^*|$, a 2-approximation. By definition, $|C| \geq |C^*|$. Thus, $|C^*| \leq |C| \leq 2|C^*|$.
Traveling Salesperson Problem (TSP)

• \textit{Instance:} \((G,c), G = (V,E), c : V \times V \rightarrow \mathbb{N}, \mathbb{N} = \{0,1,\ldots\}\)

• \textit{Objective:} Find a tour (visit each vertex once) that has minimal cost = sum of costs of edges on tour.

• We show TSP is \textbf{NP}-hard – reduce Hamiltonian Cycle (HC) to it. An HC visits each vertex once.

• \textit{Hamiltonian Cycle (HC)}

• \textit{Instance:} Graph \(G = (V,E)\)

• "Yes" \textit{Instance:} \(G\) has a HC.
Reduce HC to TSP

• Given instance $G = (V,E)$ of HC, reduce it to an instance $G' = (V,E')$ of TSP where $E' = V \times V$ and $c(u,v) = 1$ if $(u,v)$ is in $E$ and $c(u,v) = 2$ if not.

• The instance of TSP has a tour of cost $|V|$ if $G$ has an HC and cost $> |V|$ if not. Thus, TSP is not P-time unless $P = NP$. 
2-Approximation to Euclidean TSP

- In Euclidean TSP the triangle inequality holds.
  \[ c(u,v) \leq c(u,w) + c(w,v) \]

APPROX-TSP-TOUR(G,c)
1. Select any \( r \) in \( V \) as “root”.
2. Compute minimum spanning tree \( T \) of \( G \) at \( r \).
3. Let \( L = \) list of vertices in preorder tree walk of \( T \)
4. **Return** Hamiltonian Cycle \( H \) that visits vertices in the order \( L \) *without repetition*. 
2-Approximation to Euclidean TSP

- $L = \{a, c, g, c, h, c, a, b, d, b, e, f, e, b, a\}$
- $H = \{a, c, g, h, b, d, e, f, a\}$
2-Approximation to Euclidean TSP

• Cost $T$ of MST-tour of $(G, c)$ satisfies $T \leq 2|MST|$. Let $T^*$ be cost of optimal tour.

• Apply the triangle inequality

$$c(u,v) \leq c(u,w) + c(w,v)$$

to bypass repeated vertices in $L$ and form tour $H$.

• Because a spanning tree is formed by removing one edge from tour, cost of shortest tour $|MST| \leq T^*$.

• Thus $T \leq 2T^*$.

• Algorithm gives 2-approximation of Euclidean TSP.
General TSP

• In general TSP, triangle inequality doesn’t hold.
• **Theorem** Unless $P = NP$, for any $k \geq 1$ there is no $P$-time $k$-approximation algorithm for TSP.
• **Proof** As in HC to TSP reduction, assign cost 1 to edges in $G$ and $(k+1)|V|+1$ to edges not in $G$.
• Thus, if $G$ has an HC, tour length $= |V|$. If not, tour length $> (k+2)|V|$.
• If a $P$-time $k$-approximate algorithm exists, it can decide HC in $P$-time. Thus, either $P = NP$ or no $k$-approximation $P$-time algorithm exists for TSP.
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