Closure Thms:
- Regular languages are closed under \( U \), \( \ast \)
- Also closed under complement, \( L \) is regular \( \Rightarrow L^c \) is regular
- Also closed under \( \cap \), same \( F \) as \( U \), but with different \( F \)

Languages to consider:
- \( L_1 = \{ w \mid |w| \text{ is prime} \} \)
- \( L_2 = \{ w \mid w = 0^n1^n \text{ for some } n \geq 0 \} \)
- \( L_3 = \{ u \mid w \text{ has an equal } \# \text{ of } 1s \text{ and } 0s \} \)
- \( L_4 = \{ w \mid w \text{ has an equal } \# \text{ of } (01) \text{ and } (10) \} \)
- \( L_5 = \{ w \mid w = a^a \text{ for some } a \} \)
- \( L_6 = \{ w \mid w \text{ is a binary prime integer} \} \)

Thm: \( L_1 \) is not regular

Pf: We prove via contradiction.

Suppose \( L_1 \) is regular. Let \( M \) be its DFA.

Let \( m \) be the \# of states in \( M \). Let \( \eta \)
be a prime \( s.t. \), \( \eta \geq m \).

Let \( \delta \) be any \( n \)-bit string. Then, \( \delta \in \mathcal{L}(M) \).

Consider the computation history of \( \delta \) on \( M \):

\[
\delta \rightarrow \delta_0 \rightarrow \delta_1 \cdots \delta_n \rightarrow F
\]

\( 2 \eta + 1 \) states in sequence, but only \( m \) distinct states.

So, for some \( j < k \leq n \), \( \delta_j = \delta_{j+1} = \delta_{j+2} \).

Let \( W = xyz \) s.t.

1. \( |xy| \leq m \)
2. \( |y| \geq 1 \)
3. \( xy^iz \in \mathcal{L}(M) \text{ } \forall i \geq 0 \)
Then, as we have seen, $W = xyz$, and we can use 1, 2, 3.
Let $l_x = |x|$, $l_y = |y|$, $l_z = |z|$

Consider $W' = \overbrace{xyy...yz}^{\text{y repeated } l_y + l_z}$

$W' \in L(M)$ by 3

$|W'| = l_x + l_y(l_x + l_z) + l_z = (1 + l_y)(l_x + l_z)$

So $|W'|$ is not prime, $W' \notin L(M)$. Contradiction.

Thm: $L_2$ is not regular.

PF: Suppose $L_2$ is regular. Let $M$ be its DFA.
Let $m$ be the # of states in $M$. Consider $W = 0^m 1^m$

Then $W \in L_2$, and $M$ accepts $W$. Then, as we have seen, $W = xyz$ and we can use 1, 2, 3.

$W = xyz$

So, $y$ must contain only 0s. Let $|y| = l_y$

Consider $W' = xz = 0^{m-l_y} 1^m$

But, $m - l_y \neq m$ by 3. So, $|l_z = 1 |$

So, $W' \notin L_2$ and $W' \notin L(M)$. Contradiction.

Pumping Lemma: Let $L$ be a regular language, then $\exists$ some int, $m$ s.t. and $W \in L$, $|W| \geq m$, $W = xyz$ s.t.

1. $|xy| \leq m$ (m is called the pumping length)
2. $|y| \geq 1$
3. $xyz \in L \ y$ repeated $l$ times

PF: Identical to sub-proof in the proof that $L_1$ is not regular on the first page.
Thm: \( L_5 \) is not regular. Let \( m \) be its regular pumping length:

\[
W = 0^m10^m1
\]

- Then, since \(|W| \geq m\), \( W \) can be split into \( x, y, z \) s.t. the pumping lemma applies:

\[
W = xyz
\]

- First \( m \) symbols, by (1)

\[
x'y = 0^y \text{ for } y \geq 1 \text{ by (2)}
\]

\[
y = 0^1 \text{, but } W' = 0^{m-1}0^m1 \notin L \text{, contradiction.}
\]

- One approach: Pumping Lemma.
  - Pick a string to pump
  - Show that a contradiction occurs

\( 0^m10^m \) is a good candidate string.

Another approach:

- Suppose \( L_5 \) were regular, consider \( L = L_5 \setminus \{0^m1\} \)

- \( L \) must be regular \( \text{as regular languages are closed under } \setminus \)

But, \( L \neq L_5 \), which is not regular. Contradiction.

Stronger Pumping: Let \( L \) be a regular language then \( \exists \) some int. \( m \)

- S.t. \( \forall u \in L, |u| \leq m, \forall \text{ substrings } u \) or \( u' \in L, \)

\[
W = uVu_2 \rightarrow u = xyz
\]

- \( |xy| \leq m \)
- \( |y| \geq 1 \)
- \( \forall \text{ repeated } \)

\[
L \text{ times}
\]
Thm: $L_b$ is not regular

**PF:** Suppose it were, let $m$ be its pumping length.

Consider $p = a^{2^{m+1}} + 1$ (exists by Dirichlet’s Thm).

Binary rep. of $p$: \[ a \overbrace{00\ldots 01}^{m \text{ binary (in } 0s)} \]

by the strong pumping lemma, can set

$w = v_1 u v_2$, and

Then $u = x y z$, all 0s, so

$v_1 x y^k z \in L_b$

\[ = a^{0\ldots 01} \]

\[ q = a^{2^{m+k} + 1} \quad \text{let } k = p - 1 \]

\[ = a^{2^{m+2k} + 1} \]

\[ = (p-1)2^{k} + 1 \]

\[ = p \cdot 2^{k} - (2^{k-1}) \]

**Contradiction!**