Lecture 3  
Recap

DFA is 5-tuple
\[ Q : \text{set of states} \]
\[ \Sigma : \text{alphabet} \]
\[ \delta : Q \times \Sigma \to Q \text{ transition function} \]
\[ q_0 : \text{start state} \]
\[ F \subseteq Q : \text{accepting states} \]

Recall \( M \) is a DFA, \( L(M) = \{w \mid M \text{ accepts } w\} \)

A Computational History (CH) of \( M \) on input \( w = w_1w_2 \ldots w_n \)
is a sequence \( r_0, r_1 \ldots, r_n \) where \( r_0 = q_0, r_i = \delta(r_{i-1}, w_i) \text{ for } 1 \leq i \leq n \)

A CH is accepting if \( r_n \in F \)

A language \( L \) is regular if
for some DFA \( M \), \( L = L(M) \)

Closure Theorem

If \( A, B \) are regular languages, then so are

1. \( A \cup B \)  
2. \( A \circ B = \{w \mid w = w_1w_2 \text{ s.t. } w_1 \in A, w_2 \in B\} \)
3. \( A^* = \{w \mid w = w_1w_2 \ldots w_k \text{ for } k \geq 0, w_i \in A\} \)

These are called "regular operations"

Last time: showed how to union two DFAs. What about concatenation?

Hard to combine DFAs directly for concatenation

Need new model of computation

"sunny-side up" DFAs
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NFA

First, an example:

\[ L_1 = \{ w \mid w \text{ ends in } 000^3 \} \]

Let's make a non-deterministic finite automaton, \( N \) for \( L_1 \):

\[
\begin{array}{c}
q_0 & \xrightarrow{0} & q_1 & \xrightarrow{0} & q_2 & \xrightarrow{0} & q_3 \\
\end{array}
\]

Ex: \( w = 01000 \)

CHs of \( N \) on \( w \) — only one needs to accept for \( N \) to accept \( w \)

- \( q_0, q_1, q_3 \) dies (saw 1, no transition from \( q_1 \)) — rejects
- \( q_0, q_0, q_0, q_0, q_0, q_0, q_1 \) — rejects
- \( q_0, q_0, q_0, q_0, q_1, q_2, q_3 \) — accepts

So \( N \) accepts \( w \)

Another example:

\[ L_2 = \{ w \mid w \text{ is a string over } \{0, 3\}, \text{ \( |w| \) is a multiple of 2 or 3} \} \]

Let's make an NFA for \( L_2 \):

\[
\begin{array}{c}
q_0 & \xrightarrow{\varepsilon} & q_1 & \xrightarrow{\varepsilon} & q_2 & \xrightarrow{ \varepsilon, 0 } & q_3 \\
q_3 & \xrightarrow{\varepsilon} & q_5 \\
\end{array}
\]

\{ checks multiple of 2 \}

\{ checks multiple of 3 \}

Empty transition: consumes no input

Ex: \( w = 00 \)

CH of NFA on \( w \)

- \( q_0 \rightarrow q_1 \rightarrow q_2 \rightarrow q_1 \) accept
- \( q_0 \rightarrow q_3 \rightarrow q_4 \rightarrow q_5 \) reject

Formal def of NFA
A NFA $N$ is a 5-tuple, $N = (Q, \Sigma, S, q_0, F)$

- $Q$: finite set of states
- $\Sigma$: finite alphabet
- $S$: powerset of $Q$ ($\mathcal{P}(Q)$)
- $q_0$: start state
- $F \subseteq Q$: accepting states

$\delta: Q \times \Sigma \rightarrow \mathcal{P}(Q)$

$\delta(q_0, \varepsilon) \subseteq \mathcal{P}(Q)$

$\delta(q, a) \equiv \mathcal{P}(Q)$

$\delta(q, \varepsilon) \equiv \{ q \}$

A CH of an NFA $N$ on input $w$

is a sequence $y_1y_2...y_m$ s.t. $y_i \in \Sigma$ and $y_1y_2...y_m = w$

and a sequence $r_0, r_1, ..., r_m$, s.t.

$\begin{align*}
\delta(r_0, a) & \subseteq Q \\
\delta(r_i, \varepsilon) & \subseteq \mathcal{P}(Q) \\
\delta(r_i, a) & \subseteq \mathcal{P}(Q)
\end{align*}$

Note: Does not include the "dying" CHs

A CH is accepting if $r_m \in F$

$N$ accepts $w$ if $\exists$ an accepting CH of $N$ on $w$

Equivalence Theorem: Let $N$ be NFA; there exists a DFA $M$

such that $L(M) = L(N)$

Another example:

NFA for binary strings that end in 010 or end in 00

$\implies$ This would be hard to do with DFA

Now let's prove that closure thm using the equivalence thm
Proof

Let $A, B$ be regular. Let $M_A$ be DFA for $A$, $M_B$ DFA for $B$.

$M_A = (Q_A, \Sigma, \delta_A, q_{0A}, F_A)$
$M_B = (Q_B, \Sigma, \delta_B, q_{0B}, F_B)$

Want to turn into NFA for $A \cdot B$ (transitions added in purple)

Construct $N$ as follows:

$N = \delta (Q_N, \Sigma, S_N, q_{0N}, F_N)$ where

$Q_N = Q_A \cup Q_B$

$\Sigma$ same as $\delta_A(q, a) \cup \delta_B(q, a)$ if $q \in Q_A, a \in \Sigma$

$S_N(q, a) =$

$\delta_A(q, a)$ if $q \in Q_A, a \in \Sigma$

$\delta_B(q, a)$ if $q \in Q_B, a \in \Sigma$

$\delta_{A \cdot B}$ if $q \in F_A, a = \varepsilon$

$q_{0N} = q_{0A}$

$F_N = F_B$

Claim: $L(N) = A \cdot B$

Once we prove claim, it follows, by the equivalence theorem that there exists DFA $M$ s.t. $L(M) = A \cdot B$. Then $A \cdot B$ is regular.

Proof:

1. If $w \in A \cdot B$, then $w \in L(N)$ because $w = w_1 \cdot w_2$, $w_1 \in A, w_2 \in B$

   Let $y = y_1 \ldots y_{nA} \in y_1 \ldots y_{nB}$

   $w_1 \in A$

   $w_2 \in B$

   Let $r_0, r_1, \ldots r_{nA}$ be CH of $M_A$ on $w_1$ accepting, since $w_1 \in A$

   Let $r_0, r_1, \ldots r_{nB}$ be CH of $M_B$ on $w_2$ and $w_2 \in B$

   So if you concatenate CHs, you get accepting CH of $N$ on $w$

2. Let $w \in L(N)$, then $w = w_1 \cdot w_2$ s.t. $w_1 \in A, w_2 \in B$

   Let CH of $N$ on $w$ be $y = y_1, \ldots, y_{m}$

   $r = r_0, \ldots, r_m$

   $y_i = \varepsilon$ in $M$

   (see diagram)