CS 1010 Lecture 17
More on NP, co-NP, NP-complete languages

Last time: SAT and 3-SAT are NP-complete

Thm: Let A, B, C be languages.
If A ≤_P B and B ≤_P C, then A ≤_P C.

Pf: Let f_1 be the polynomial function s.t. x ∈ A iff f_1(x) ∈ B. Similarly, f_2 is the polynomial function s.t. x ∈ B iff f_2(x) ∈ C. f_1, f_2 exist due to the given polynomial reducibility.
Then let f(x) = f_2(f_1(x)).

X ∈ A ↔ f_1(x) ∈ B ↔ f_2(f_1(x)) ∈ C ↔ f(x) ∈ C, as needed.

Runtime: f is polynomial.

1. Compute y = f_1(x), need p_1(|x|) steps, so M ≤_p p_1(|x|).
2. Compute z = f_2(y), need p_2(|x|) steps, or ≤ p_2(p_1(|x|)) steps, still polynomial.

Corollary: If A, B ∈ NP and A is NP-complete and A ≤_P B, then B is NP-complete.

Pf: With A ∈ NP, L ≤_P B. Let L ∈ NP, L ≤_P A b/c A is NP-complete. Then, by thm above, L ≤_P B.

Another Corollary: If A ∈ P is NP-complete, then \( P = NP \).

Pf: Let L ∊ NP, to decide L in polynomial:
on input x:
1. Compute y = f(x) where f is s.t.
   x ∈ L ↔ f(x) ∈ A (\( \exists b/c \ A ∈ NP \)-complete)
2. Decide in polynomial of y ∈ A (\( \exists b/c \ A ∈ P \))
**CO NP**

\[ \text{CO NP} = \{ L \mid L \in \text{NP} \} \]

**Ex: unsatisfiability**

\[ \text{SAT} = \{ \emptyset \} \text{ no assignment can satisfy} \]

**Current picture:**

\[ P = \text{NP} \]

\[ N = \text{CO NP} \]

\[ \text{P} = \text{NP} \geq \text{open} \]

\[ \text{NP} = \text{CO NP} \geq \text{open} \]

**Def:** \( A \) is \( \text{NP-complete} \) iff

1. \( A \in \text{NP} \)
2. \( A \) is \( \text{NP-hard} \) (\( \forall L \in \text{NP}, L \leq_p A \))

**Thm:** \( A \) is \( \text{NP-complete} \) iff \( A^c \) is \( \text{CO NP-complete} \)

**Pf:** Let \( L \leq_p \text{CO NP} \), wts \( L \leq_p A^c \).

\( L \leq_p A \) b/c \( A \) is \( \text{NP-complete} \). So, \( f \)

Polynomial \( f \) s.t. \( x \in L \iff f(x) \notin A \)

\( x \in L \iff f(x) \notin A^c \)

So, \( L \leq_p A^c \), similar for other direction

**More NP-complete:**

\( \text{CLIQUE} = \{ \langle G, K \rangle \mid G \text{ is a graph with a } K \text{-clique} \} \)

**Thm:** \( \text{CLIQUE} \) is \( \text{NP-complete} \)

**Pf:** \( \text{CLIQUE} \in \text{NP} \) (shown in past class)

\( \text{WTS: } 3\text{SAT} \leq_p \text{CLIQUE} \)

**Reduction:** On input \( \langle B \rangle \), construct

\[ G = (V, E) \]

\[ V = \{ V_i \mid \text{each } V_i \text{ corresponds to } y_i \} \]

\[ E = \{ \{ V_i, V_j \} \mid B_i \land B_j \text{ is true} \} \]
\[ E = \{(V_i, j, V_{i', j}) \mid i \neq i' \text{ and } j \neq j'\} \]

Output \((G, m)\) (\(m\) is # of clauses)

Analysis:

- Runtime: Polynomial. Making a polynomial
  # of vertices, edges, in constant time each

- Correctness: 2 directions

1. \((\emptyset) \in 3SAT \Rightarrow (G, m) \notin \text{Clique b/c}
   \begin{align*}
   &\text{let } a_1, \ldots, a_n \text{ be an assignment to } x_1, \ldots, x_n \text{ that satisfies } \emptyset. \\
   &U = \{ u_i \mid u_i \in \{V_i, 1, V_i, 2, V_i, 3\} \text{ and } u_i \text{ corresponds to } y_i, j \text{ that is satisfied by } a_1, \ldots, a_n \} \\
   &\forall i, u_i \text{ exists.} \\
   &|U| = m. U \text{ is an } m \text{-clique b/c } U \text{, } u_i, j \text{, } i \neq i' \text{ correspond to } U \text{, } y_i, j, y_i, j', 60 \text{th s.} \text{ satisfied by } a_1, \ldots, a_n \text{. So } (y_i, j, y_i, j') \in E(G) \\
   \end{align*}

2. \((G, m) \notin \text{Clique} \Rightarrow (\emptyset) \in 3SAT
   \begin{align*}
   &\text{let } U \text{ be an } m \text{-clique in } G. \\
   &U = \{ u_i \mid 1 \leq i \leq m, u_i \in \{V_i, 1, V_i, 2, V_i, 3\} \} \\
   &b/c \text{ no edges inside triple.} \\
   &\text{Let } a_k = \begin{cases} 
   1 & \text{if } y_i, j = x_k \text{ for some } V_i, j \in U \\
   0 & \text{otherwise} \end{cases} \\
   &\text{Resulting assignment satisfies } \emptyset
   \end{align*}

Corollary: vertex cover + independent set are NP-complete.
3-colorability = \{ (G) \mid G \text{ is 3-colorable} \}

A graph is \( k \)-colorable if its vertices can be partitioned into \( V_1, \ldots, V_k \) such that \( (u, v) \notin E \) if \( u, v \) are in different partitions.

Thm 3-colorability is NP-complete.

Pf: (1) NP: Exercise
(2) 3SAT \leq P 3-colorability
   colors are true, false, and neutral

For each clause add the following:

true vertex from \( l \)

false vertex from \( l \)