Lecture 11:
undecidability, Rice’s Thm

Recap:

• $\text{Atm}$ is undecidable

Today’s examples:

• $\text{AnnAtm} = \{<M> | \text{Anna} \in L(M)\}$

• $\text{RegularAtm} = \{<M> | L(M) \text{ is regular}\}$

$\text{Atm} = \{<M, w> | \text{TM } M \text{ accepts } w\}$

Def. A language $L$ is decidable if

1. $L = L(M)$
2. $M$ is a decider. It halts on every input

$\text{AnnAtm}$

Is it Turing recognizable?

Consider a TM $R$:

```
  on input <M>
  - run $M$ on input "anna"
  - if $M$ accepts, accept; else, reject.
```

Analysis:

if $<M> \in \text{AnnAtm}$, then $M$ accepts "anna", so $R$ accepts $<M>$.

if $<M> \notin \text{AnnAtm}$, then either:

- $M$ does not halt on input "anna", so $R$ does not accept $<M>$, so $R$ does not accept $<M>$.

- $M$ rejects input "anna", so $R$ rejects input $<M>$.

So, $R$ accepts elements in $\text{AnnAtm}$ and does not accept elements not in $\text{AnnAtm}$, as needed.

Is it decidable?

Suppose for the sake of contradiction that it is, and we have a decider $D_{\text{AnnAtm}}$.

Consider a decider $D$ for $\text{Atm}$:

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  on input $<M, w>$
  - compute $<M'>$, a TM s.t. $<M'> \in \text{AnnAtm}$ iff $<M, v'> \in \text{Atm}$
    - run $D_{\text{AnnAtm}}$ ($<M'>$), if it accepts, accept.
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We now define $M'$:

"On input $S$,
- Run $M$ on $W$
- If it accepts, accept. Else, reject."

Analysis of $D$:
Suppose $\langle M, W \rangle \in \text{ATM}$. Then $\langle M' \rangle \in \text{ANM}$.
Then $D'_{\text{ANM}}$ accepts $\langle M' \rangle$, then $D$ accepts $\langle M, W \rangle$, as needed.

Suppose $\langle M, W \rangle \in \text{ATM}$. then $\langle M' \rangle \in \text{ANM}$.
Then $D'_{\text{ANM}}$ rejects $\langle M' \rangle$, then $D$ rejects $\langle M, W \rangle$ as needed.

So, $D$ is a decider for $\text{ATM}$, but $\text{ATM}$ is undecidable. This is a contradiction, so $\text{ANM}$ is undecidable.

Regular, $\text{R}$ is it decidable?
In use same proof structure as for $\text{ANM}$, but we redefine $M$:

"On input $S$
- If $S = 0^n 1^n$, accept
- Else, run $M$ on $W$. If $M$ accepts, accept. Else, reject."

First, we show that $M$ satisfies $\langle M' \rangle \in \text{Regular}$ iff $\langle M, W \rangle \in \text{ATM}$.

Suppose $\langle M, W \rangle \in \text{ATM}$. Then $M$ accepts $W$, so $M'$ accepts all strings $\Rightarrow L(M') = \Sigma^*$, which is regular, so $\langle M' \rangle \in \text{Regular}$.

Suppose $\langle M, W \rangle \in \text{ATM}$. Then $M$ does not accept $W$, so $M'$ only accepts $S = 0^n 1^n$, so $L(M') = 0^n 1^n$, which is not regular, so $\langle M' \rangle \notin \text{Regular}$, as needed.

Analysis of $D$ is identical, contradiction follows, so Regular is not decidable.
Rice's Thm

Let \( P \) be a set of TMs s.t.
- (0) non-trivial: \( \exists M_1, M_2 \) s.t. \( M_1 \in P, M_2 \notin P \)
- (1) \( P \) is a property of the language:
  - if \( L(M_1) = L(M_2) \), either \( M_1, M_2 \in P \) or \( M_1, M_2 \notin P \)

Then \( L = \{ \langle M \rangle | M \in P \} \) is undecidable.

PF: Let \( M_{\text{rej}} \) be a TM that always rejects. Suppose \( M_{\text{rej}} \in P \). Let \( M \in P \).

Suppose \( L \) has a decider \( D_L \). Consider a decider \( D \) for \( ATM \)
- On input \( \langle M, w \rangle \), construct \( \langle M' \rangle \), a TM such that:
  - \( M' \in P \) iff \( \langle M, w \rangle \notin ATM \)
  - Run \( D_L \langle M' \rangle \). Accept if it accepts, reject if it rejects.

Now we show \( M' \) satisfies (0).
- Suppose \( \langle M, w \rangle \notin ATM \), then \( L(M') = L(M) \). Since \( M \in P \), so is \( M' \) by (1).
- Suppose \( \langle M, w \rangle \in ATM \), then \( L(M') = L(M_{\text{rej}}) \). Since \( M_{\text{rej}} \in P \), so is \( M' \) by (1).

Analysis is nearly identical to that of \( ATM \).

Suppose \( M_{\text{rej}} \in P \). Then consider \( P \circ M_{\text{rej}} \in P \), so \( L = \{ \langle M \rangle | M \in P \} \) is undecidable \( \Rightarrow L \) is undecidable.
Examples of undecidable languages by Rice's Thm:
- $\text{Empty}_{TM} = \{ \langle M \rangle \mid L(M) = \emptyset \}$
- $\text{Finite}_{TM} = \{ \langle M \rangle \mid M \text{ accepts a finite # of inputs} \}$

Examples of undecidable languages where Rice's thm does not apply:
- $\text{EQ}_{TM} = \{ \langle M_1, M_2 \rangle \mid L(M_1) = L(M_2) \}$
- $L = \{ \langle M \rangle \mid \exists M' \text{ s.t. } M' \text{ has same # of states as } M, \text{ but } M' \neq M \text{ and } L(M') = L(M) \}$

**Thm: EQ$_{TM}$ is undecidable**

**Proof:** Suppose DEQ is a decider for EQ$_{TM}$, then here is a decider $D$ for ATM:

```
D(M, w):
    - Construct $M_1, M_2$ s.t. $L(M_1) = L(M_2)$ if $M$ accepts $w$.
    - Run DEQ($\langle M_1, M_2 \rangle$), output what it outputs.
```

$M_1$: on input $s$
```
run $M$ on $s$
accept if $M$ accepts
reject otherwise
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$M_2$: on input $s$
```
run $M$ on $w$
accept if $M$ accepts
reject otherwise
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