CSCI 1010  Theory of Computation

HW1

Due: September 12, 2019

Reminder: Lab problems are graded and must be submitted along with homework when lab writeups are due. Your name should not appear anywhere on your handin; each individual page of the homework should include your Banner ID only. For your digital submission, each page should include work for only one problem (i.e., make a new page/new pages for each problem).

While collaboration is encouraged in this class, please remember not to take away notes from any labs or collaboration sessions.

Please monitor Piazza, as we will post clarifications of questions there. Hand in your solutions by 10:20am to the CSCI 1010 bin on the second floor of the CIT. Late homeworks are not accepted.

Problem 1

Let $n$ be a composite positive integer. Prove that $n$ has a prime divisor less than or equal to $\sqrt{n}$.

Problem 2

Let $\beta(n, k)$ be the number of ways to place $n$ labeled ingredients into exactly $k$ labeled bins. Here, “exactly” means none of the bins should be empty.

a. Give expressions for $\beta(n, k)$ for $k = 1$, $k = 2$, $k = 3$, and $k = 4$. Give a brief explanation for each expression.

b. It can be shown by induction that in general, $\beta(n, k) = \sum_{j=1}^{k} (-1)^{k-j} \binom{k}{j} j^n$.
   Justify why the sum $\sum_{j=0}^{n+1} (-1)^{n+1-j} \binom{n+1}{j} j^n$ should equal zero.

Problem 3

Gordon Ramsay wants to make sure all of the USELESS COMPETITORS in Hell’s Kitchen know the way around the studio so they CAN’T MESS ANYTHING UP MORE THAN THEY ALREADY HAVE.
He decides to use graph theory to prove that it is easy for competitors to get back to their DisGusTinG cooking if they get lost.

Let $G$ be a graph with $n$ nodes where $n \geq 3$. For each pair of nodes $(u, v) \in G$, there is a directed edge either $u \rightarrow v$ or $v \rightarrow u$. Each node represents a room in the studio, and each edge represents a way to get from one room to another (remember, these are directed! Fox enforces strict access control). Prove that $G$ has a node $x$ called the kitchen such that there is a directed path with length at most two from every node other than $x$ in $G$ to the kitchen.

Hint: There is an inductive proof that has the base case of a graph with three nodes. Consider all situations that could arise when you add in a fourth node.

Problem 4

The chefs at the 1010 kitchen have been planning to open a pop-up restaurant, but they are having trouble making decisions. The (secret) celebrity chef in charge of the kitchen has stepped in and given a list of $n$ dishes that she would want to include on the menu. However, this esteemed chef is definitely much less picky than Gordon Ramsay, so she will be satisfied if only one of her proposed menu items makes it onto the final lineup. Since the 1010 kitchen managers love circuits, they have decided to devise a Boolean function to model this chef’s satisfaction with the final menu.

Consider the Boolean function

$$f(x_1, x_2, \ldots, x_n) = x_1 \lor x_2 \lor \cdots \lor x_n.$$

The number of gates required to construct a circuit that computes this function for a fixed $n$ is dependent on the number of inputs allowed to each OR gate in the circuit.

a. Show that the number of 2-input OR gates required to construct a circuit that computes $f$ is $n - 1$ and the depth of such a circuit is at least $\lceil \log_2 n \rceil$.

b. Find and justify a similar formula for the size and depth of a circuit constructed using $r$-input OR gates for an arbitrary (but fixed) $r$. 