Problem 1

The Pinary Tree Company is a competing forestry organization in Yew Nork. Its leaders, Douglas and Connie Ferr, are investing in research for their circuitree department. They’ve hired you to prove some preliminary results about circuit satisfiability.

Consider a circuit with $n$ input wires and one output wire. Let the set of the circuit’s input wires be $I_n = \{w_1, w_2, \ldots, w_n\}$ where $w_i$ represents the $i$th wire. Consider some subset of input wires $t \subseteq I_n$. We say that the subset satisfies the circuit if the circuit evaluates to $\text{True}$ when all input wires in $t$ are set to $\text{True}$ and all input wires not in $t$ are set to $\text{False}$.

Let $T_n$ be the set of all $t \subseteq I_n$ such that $t$ satisfies the circuit. Prove that for any $n \geq 2$, there exists a circuit where:

a. $|T_n| = 1$

b. $|T_n| = 2^n - 1$

c. $|T_n| = 2^n$

d. $|T_n| = 2^{n-1}$

e. $|T_n| = n$

You do not need to use induction or draw circuit diagrams for this problem.
Problem 2

You are playing a variant of the well-known game Snake, and start to notice a pattern in the snake’s coloring. A snake contains squares arranged in a straight line (unlike regular Snake, the snake does not bend), and each square is colored either dark purple or light blue. In addition, the first square (the head) must be dark purple, and the last square must be light blue. The figure below shows an example of a snake.

Prove that, in any snake, there are an odd number of places where consecutive squares are colored differently. For example, in the figure, there are three pairs of consecutive squares where one square is colored dark purple and the other is colored light blue.

(Hint: it might help to think of the snake as a graph where the squares are vertices and consecutive squares are joined by an edge.)

Problem 3

Let $n$ be a composite positive integer. Prove that $n$ has a prime divisor less than or equal to $\sqrt{n}$.

Problem 4

1. When it was first discovered that there are computational problems that cannot be solved by computers, it was a very surprising result. In this problem, we will do a high-level argument of a related result, due to Gödel. Gödel showed that there exist true mathematical statements that cannot be proven. Instead of using mathematical statements, we will simply use English sentences.
This is the sentence we are trying to prove:

*There exists a true sentence for which there is no proof.*

We will argue this by contradiction.\(^1\) That is, we will assume the following statement, then obtain a contradiction:

*Assumption: All true sentences have proofs.*

You may assume that any statement for which there exists a proof is true. (Hint: Examine the sentence \(S = \text{“No proof exists for this sentence.”} \) Show that given our assumption, if \(S\) is true or false, we obtain a contradiction.)

2. In class we looked at the Halting Problem, the problem of determining whether a program \(P\) will terminate on an input \(X\). We wanted to know whether a program \text{HP Solver} that solves the Halting Problem exists:

\[
\text{HP Solver:} \\
\text{Input: a program } P, \text{ an input } X \\
\text{Output: } \text{Yes if } P \text{ terminates on } X, \text{ No if } P \text{ runs forever}
\]

We concluded that \text{HP Solver} cannot exist, meaning that no program can solve the halting problem. We will now look at another problem that cannot be solved by computers, the problem of deciding the truth or falsity of a sentence. We want to know whether a program \text{Truth Solver} exists:

\[
\text{Truth Solver:} \\
\text{Input: a sentence } S \\
\text{Output: } \text{True if } S \text{ is true, False if } S \text{ is false}
\]

Argue that \text{Truth Solver} cannot exist, by first showing that if we had a \text{Truth Solver}, we could use it to build an \text{HP Solver}. Then argue that since \text{HP Solver} cannot exist, \text{Truth Solver} cannot exist. (Hint: \text{HP Solver} will use \text{Truth Solver} as a subroutine. What kind of statement should it give as input?)

\(^1\) We are being informal here, because we have not given a mathematical formalization of the statement we are trying to prove.