Quiz

- Restate equations

\[ v_0 = v_0^* \]
\[ v_1 = \alpha_{01} v_0^* + v_1 \]
\[ v_2 = \alpha_{02} v_0^* + \alpha_{12} v_1^* + v_2 \]
\[ v_3 = \alpha_{03} v_0^* + \alpha_{13} v_1^* + \alpha_{23} v_2^* + v_3^* \]

as a single matrix equation: (some matrix) = (some other matrix) (yet another matrix)

- What is special about the two matrices on the right-hand side?

- Let \( v_0^*, v_1^*, v_2^*, v_3^* \) be mutually orthogonal vectors. Assume in addition that they are nonzero. You are to show that they are linearly independent. That is, if \( \alpha_0, \alpha_1, \alpha_2, \alpha_3 \) satisfy the equation

\[ 0 = \alpha_0 v_0^* + \alpha_1 v_1^* + \alpha_2 v_2^* + \alpha_3 v_3^* \]

then \( \alpha_0 = 0, \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0 \).

For purposes of this proof, it suffices to show that \( \alpha_0 = 0 \).
Matrix form for orthogonalize

For project orthogonal, we had

\[
\begin{bmatrix}
\mathbf{b} \\
\mathbf{v}_0 & \cdots & \mathbf{v}_n & \mathbf{b}^\perp
\end{bmatrix}
= \begin{bmatrix}
\alpha_0 \\
\vdots \\
\alpha_n \\
1
\end{bmatrix}
\]

For orthogonalize, we have

\[
\begin{bmatrix}
\mathbf{v}_0 \\
\mathbf{v}_1 \\
\mathbf{v}_2 \\
\mathbf{v}_3
\end{bmatrix}
= \begin{bmatrix}
\mathbf{v}_0^* \\
\mathbf{v}_1^* \\
\mathbf{v}_2^* \\
\mathbf{v}_3^*
\end{bmatrix}
\]

The two matrices on the right are special:

- I Columns of first one are mutually orthogonal.
- I Second is upper triangular.

We will use these properties in algorithms...
Proposition: Mutually orthogonal nonzero vectors are linearly independent.

Proof: Let $v_0^*, v_1^*, \ldots, v_n^*$ be mutually orthogonal nonzero vectors. Suppose $\alpha_0, \alpha_1, \ldots, \alpha_n$ are coefficients such that

$$0 = \alpha_0 v_0^* + \alpha_1 v_1^* + \cdots + \alpha_n v_n^*$$

We must show that therefore the coefficients are all zero.
To show that $\alpha_0$ is zero, take inner product with $v_0^*$ on both sides:

$$\langle v_0^*, 0 \rangle = \langle v_0^*, \alpha_0 v_0^* + \alpha_1 v_1^* + \cdots + \alpha_n v_n^* \rangle$$
$$= \alpha_0 \langle v_0^*, v_0^* \rangle + \alpha_1 \langle v_0^*, v_1^* \rangle + \cdots + \alpha_n \langle v_0^*, v_n^* \rangle$$
$$= \alpha_0 \|v_0^*\|^2 + \alpha_1 0 + \cdots + \alpha_n 0$$
$$= \alpha_0 \|v_0^*\|^2$$

The inner product $\langle v_0^*, 0 \rangle$ is zero, so $\alpha_0 \|v_0^*\|^2 = 0$. Since $v_0^*$ is nonzero, its norm is nonzero, so the only solution is $\alpha_0 = 0$.
Can similarly show that $\alpha_1 = \cdots = \alpha_n = 0$. QED
Computing a basis

**Proposition:** Mutually orthogonal nonzero vectors are linearly independent.

What happens if we call the orthogonalize procedure on a list $vlist = [v_0, \ldots, v_n]$ of vectors that are linearly dependent?

$\dim \text{Span} \{v_0, \ldots, v_n\} < n + 1.$

$\text{orthogonalize}([v_0, \ldots, v_n])$ returns $[v_0^*, \ldots, v_n^*]$.

The vectors $v_0^*, \ldots, v_n^*$ are mutually orthogonal.

They can't be linearly independent since they span a space of dimension less than $n + 1$.

Therefore some of them must be zero vectors.

Leaving out the zero vectors does not change the space spanned.

Let $S$ be the subset of $\{v_0^*, \ldots, v_n^*\}$ consisting of nonzero vectors.

$\text{Span} \ S = \text{Span} \ \{v_0^*, \ldots, v_n^*\} = \text{Span} \ \{v_0, \ldots, v_n\}$

Proposition implies that $S$ is linearly independent.

Thus $S$ is a basis for $\text{Span} \ \{v_0, \ldots, v_n\}$. 
Computing a basis

Therefore in principle the following algorithm computes a basis for \( \text{Span} \{ \mathbf{v}_0, \ldots, \mathbf{v}_n \} \):

```python
def find_basis([\mathbf{v}_0, \ldots, \mathbf{v}_n]):
    "Return the list of nonzero starred vectors."
    [\mathbf{v}^*_0, \ldots, \mathbf{v}^*_n] = orthogonalize([\mathbf{v}_0, \ldots, \mathbf{v}_n])
    return [\mathbf{v}^*_i for \mathbf{v}^*_i in [\mathbf{v}^*_0, \ldots, \mathbf{v}^*_n] if \mathbf{v}^*_i is not the zero vector]
```

Example:
Suppose \( \text{orthogonalize}([\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6]) \) returns \([\mathbf{v}^*_0, \mathbf{v}^*_1, \mathbf{v}^*_2, \mathbf{v}^*_3, \mathbf{v}^*_4, \mathbf{v}^*_5, \mathbf{v}^*_6]\) and the vectors \( \mathbf{v}^*_2, \mathbf{v}^*_4, \) and \( \mathbf{v}^*_5 \) are zero.
Then the remaining output vectors \( \mathbf{v}^*_0, \mathbf{v}^*_1, \mathbf{v}^*_3, \mathbf{v}^*_6 \) form a basis for \( \text{Span} \{ \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6 \} \).

Recall **Lemma**: Every finite set \( T \) of vectors contains a subset \( S \) that is a basis for \( \text{Span} \ T \).

What about finding a subset of \( \mathbf{v}_0, \ldots, \mathbf{v}_n \) that is a basis?

**Proposed algorithm:**
```python
def find_subset_basis([\mathbf{v}_0, \ldots, \mathbf{v}_n]):
    "Return the list of original vectors that correspond to nonzero starred vectors."
    [\mathbf{v}_0, \ldots, \mathbf{v}_n] = orthogonalize([\mathbf{v}_0, \ldots, \mathbf{v}_n])
    return [\mathbf{v}_i for i in {0, \ldots, n} if \mathbf{v}^*_i is not the zero vector]
```
Computing a basis

Therefore in principle the following algorithm computes a basis for \( \text{Span } \{v_0, \ldots, v_n\} \):

```python
def find_basis([v_0, \ldots, v_n]):
    "Return the list of nonzero starred vectors."
    [v^*_0, \ldots, v^*_n] = orthogonalize([v_0, \ldots, v_n])
    return [v^* for v^* in [v^*_0, \ldots, v^*_n] if v^* is not the zero vector]
```

Recall **Lemma**: Every finite set \( T \) of vectors contains a subset \( S \) that is a basis for \( \text{Span } T \).

What about finding a subset of \( v_0, \ldots, v_n \) that is a basis?

**Proposed algorithm**:

```python
def find_subset_basis([v_0, \ldots, v_n]):
    "Return the list of original vectors that correspond to nonzero starred vectors."
    [v^*_0, \ldots, v^*_n] = orthogonalize([v_0, \ldots, v_n])
    return [v_i for i in \{0, \ldots, n\} if v^*_i is not the zero vector]
```

Is this correct?
Correctness of find_subset_basis

```python
def find_subset_basis([v0, ..., vn]):
    [v0^*, ..., vn^*] = orthogonalize([v0, ..., vn])
    return [vi for i in {0, ..., n} if vi^* is not the zero vector]

def orthogonalize(vlist):
    vstarlist = []
    for v in vlist:
        vstarlist.append(project_orthogonal(v, vstarlist))
    return vstarlist
```

Example: `orthogonalize([v0, v1, v2, v3, v4, v5, v6])` returns `[v0^*, v1^*, v2^*, v3^*, v4^*, v5^*, v6^*]`

Suppose v2^*, v4^*, and v5^* are zero vectors.

In iteration 3 iteration of orthogonalize, `project_orthogonal(v3, [v0^*, v1^*, v2^*])` computes v3^*:
- subtract projection of v3 along v0^*,
- subtract projection along v1^*,
- subtract projection along v2^*—but since v2^* = 0, the projection is the zero vector

Result is the same as `project_orthogonal(v3, [v0^*, v1^*])`. Zero starred vectors are ignored.

Thus `orthogonalize([v0, v1, v3, v6])` would return `[v0^*, v1^*, v3^*, v6^*]`.

Since `[v0^*, v1^*, v3^*, v6^*]` is a basis for V = Span \{v0, v1, v2, v3, v4, v5, v6\}
Correctness of find_subset_basis

```python
def find_subset_basis([v0, ..., vn]):
    [v^*, ..., v^*_n] = orthogonalize([v0, ..., vn])
    Return [v_i for i in {0, ..., n} if v^*_i is not the zero vector]
```

Suppose v^*_2, v^*_4, and v^*_5 are zero vectors. In iteration 3 iteration of orthogonalize, \( \text{project}_\text{orthogonal}(v_3, [v^*_0, v^*_1, v^*_2]) \) computes \( v^*_3 \):

- subtract projection of \( v_3 \) along \( v^*_0 \),
- subtract projection along \( v^*_1 \),
- subtract projection along \( v^*_2 \)—but since \( v^*_2 = 0 \), the projection is the zero vector

Result is the same as \( \text{project}_\text{orthogonal}(v_3, [v^*_0, v^*_1]) \). Zero starred vectors are ignored. Thus \( \text{orthogonalize}([v_0, v_1, v_3, v_6]) \) would return \( [v^*_0, v^*_1, v^*_3, v^*_6] \).

Since \( [v^*_0, v^*_1, v^*_3, v^*_6] \) is a basis for \( \mathcal{V} = \text{Span} \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\} \) and \( [v_0, v_1, v_3, v_6] \) spans the same space, and has the same cardinality.

\( [v_0, v_1, v_3, v_6] \) is also a basis for \( \mathcal{V} \).
Correctness of find_subset_basis

Another way to justify find_subset_basis...

Here’s the matrix equation expressing original vectors in terms of starred vectors:

\[
\begin{bmatrix}
\mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n
\end{bmatrix}
= \begin{bmatrix}
\mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_2^* & \cdots & \mathbf{v}_n^*
\end{bmatrix}
\begin{bmatrix}
1 & \alpha_{01} & \alpha_{02} & \cdots & \alpha_{0n} \\
1 & \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\
1 & \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & & & & 1
\end{bmatrix}
\]
Correctness of find_subset_basis

Let $V = \text{Span} \{ \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6 \}$. Suppose $\mathbf{v}^*_2$, $\mathbf{v}^*_4$, and $\mathbf{v}^*_5$ are zero vectors.

\[
\begin{bmatrix}
\mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 & \mathbf{v}_5 & \mathbf{v}_6 \\

\end{bmatrix}
\]

\[
= \begin{bmatrix}
\mathbf{v}^*_0 & \mathbf{v}^*_1 & \mathbf{v}^*_2 & \mathbf{v}^*_3 & \mathbf{v}^*_4 & \mathbf{v}^*_5 & \mathbf{v}^*_6 \\

\end{bmatrix}
\]

Delete zero columns and the corresponding rows of the triangular matrix. Shows $\text{Span} \{ \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6 \} \subseteq \text{Span} \{ \mathbf{v}^*_0, \mathbf{v}^*_1, \mathbf{v}^*_3, \mathbf{v}^*_6 \}$

so $\{ \mathbf{v}^*_0, \mathbf{v}^*_1, \mathbf{v}^*_3, \mathbf{v}^*_6 \}$ is a basis for $V$

Delete corresponding original columns $\mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_5$.

Resulting triangular matrix is invertible. Move it to other side. Shows $\text{Span} \{ \mathbf{v}^*_0, \mathbf{v}^*_1, \mathbf{v}^*_2, \mathbf{v}^*_6 \} \subseteq \text{Span} \{ \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6 \}$ so $\{ \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6 \}$ is basis for $V$. QED
Correctness of find_subset_basis

Let $\mathcal{V} = \text{Span} \{ \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6 \}$.
Suppose $\mathbf{v}_2^*, \mathbf{v}_4^*$, and $\mathbf{v}_5^*$ are zero vectors.

$\begin{bmatrix}
\mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_3 & \mathbf{v}_6 \\
\end{bmatrix}$

$= \begin{bmatrix}
\mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_3^* & \mathbf{v}_6^* \\
\end{bmatrix}$

Delete zero columns and the corresponding rows of the triangular matrix. Shows $\text{Span} \{ \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6 \} \subseteq \text{Span} \{ \mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^* \}$
so $\{ \mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^* \}$ is a basis for $\mathcal{V}$.
Delete corresponding original columns $\mathbf{v}_2$, $\mathbf{v}_4$, $\mathbf{v}_5$.
Resulting triangular matrix is invertible. Move it to other side.
Shows $\text{Span} \{ \mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_2^*, \mathbf{v}_6^* \} \subseteq \text{Span} \{ \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6 \}$ so $\{ \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6 \}$ is basis for $\mathcal{V}$.

QED
Correctness of find_subset_basis

\[
\begin{bmatrix}
  \mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_3 & \mathbf{v}_6 \\
\end{bmatrix}
\]

Let \( \mathcal{V} = \text{Span} \{ \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6 \} \).

Suppose \( \mathbf{v}_2^*, \mathbf{v}_4^*, \) and \( \mathbf{v}_5^* \) are zero vectors.

\[
= \begin{bmatrix}
  \mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_3^* & \mathbf{v}_6^* \\
\end{bmatrix}
\]

Delete zero columns and the corresponding rows of the triangular matrix. Shows \( \text{Span} \{ \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6 \} \subseteq \text{Span} \{ \mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^* \} \)
so \( \{ \mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^* \} \) is a basis for \( \mathcal{V} \)

Delete corresponding original columns \( \mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_5 \).

Resulting triangular matrix is invertible. Move it to other side.

Shows \( \text{Span} \{ \mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_2^*, \mathbf{v}_6^* \} \subseteq \text{Span} \{ \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6 \} \) so \( \{ \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6 \} \) is basis for \( \mathcal{V} \).

QED
Correctness of find_subset_basis

\[
\begin{bmatrix}
\mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_3 & \mathbf{v}_6
\end{bmatrix}
\]

Let \( \mathcal{V} = \text{Span } \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\} \).

Suppose \( \mathbf{v}_2^*, \mathbf{v}_4^*, \text{ and } \mathbf{v}_5^* \) are zero vectors.

\[
= \begin{bmatrix}
\mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_3^* & \mathbf{v}_6^*
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} & \alpha_{05} & \alpha_{06}
1 & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16}
1 & \alpha_{24} & \alpha_{35} & \alpha_{36}
1
\end{bmatrix}
\]

Delete zero columns and the corresponding rows of the triangular matrix. Shows \( \text{Span } \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\} \subseteq \text{Span } \{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*\} \)

so \( \{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*\} \) is a basis for \( \mathcal{V} \)

Delete corresponding original columns \( \mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_5 \).

Resulting triangular matrix is invertible. Move it to other side.

Shows \( \text{Span } \{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_2^*, \mathbf{v}_6^*\} \subseteq \text{Span } \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6\} \) so \( \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6\} \) is basis for \( \mathcal{V} \). QED
Correctness of find_subset_basis

\[
[v_0 \ v_1 \ v_3 \ v_6]
\]

Let \( \mathcal{V} = \text{Span} \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\} \).

Suppose \( v^*_2, v^*_4, \) and \( v^*_5 \) are zero vectors.

\[
= \begin{bmatrix}
v^*_0 & v^*_1 & v^*_3 & v^*_6
\end{bmatrix}
\]

\[
= \begin{bmatrix}
v^*_0 & v^*_1 & v^*_3 & v^*_6
\end{bmatrix}
\]

Delete zero columns and the corresponding rows of the triangular matrix. Shows

\( \text{Span} \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\} \subseteq \text{Span} \{v^*_0, v^*_1, v^*_3, v^*_6\} \)

so \( \{v^*_0, v^*_1, v^*_3, v^*_6\} \) is a basis for \( \mathcal{V} \).

Delete corresponding original columns \( v_2, v_4, v_5 \).

Resulting triangular matrix is invertible. Move it to other side.

\[
\begin{bmatrix}
1 & \alpha_{01} & \alpha_{03} & \alpha_{06}
1 & \alpha_{13} & \alpha_{16}
1 & \alpha_{36}
\end{bmatrix}
\]

Shows \( \text{Span} \{v^*_0, v^*_1, v^*_2, v^*_6\} \subseteq \text{Span} \{v_0, v_1, v_3, v_6\} \) so \( \{v_0, v_1, v_3, v_6\} \) is basis for \( \mathcal{V} \).

QED
Correctness of find_subset_basis

Let $\mathcal{V} = \text{Span} \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$. Suppose $\mathbf{v}_2^*, \mathbf{v}_4^*$, and $\mathbf{v}_5^*$ are zero vectors.

\[
\begin{bmatrix}
\mathbf{v}_0 & \mathbf{v}_1 & \mathbf{v}_3 & \mathbf{v}_6 \\
\end{bmatrix}
= \begin{bmatrix}
\mathbf{v}_0^* & \mathbf{v}_1^* & \mathbf{v}_3^* & \mathbf{v}_6^* \\
\end{bmatrix}
\]

Delete zero columns and the corresponding rows of the triangular matrix. Shows $\text{Span} \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\} \subseteq \text{Span} \{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*\}$

so $\{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_3^*, \mathbf{v}_6^*\}$ is a basis for $\mathcal{V}$

Delete corresponding original columns $\mathbf{v}_2, \mathbf{v}_4, \mathbf{v}_5$.

Resulting triangular matrix is invertible. Move it to other side.

Shows $\text{Span} \{\mathbf{v}_0^*, \mathbf{v}_1^*, \mathbf{v}_2^*, \mathbf{v}_6^*\} \subseteq \text{Span} \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6\}$ so $\{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_6\}$ is basis for $\mathcal{V}$. QED
Correctness of find_subset_basis

\[
\begin{bmatrix}
v_0 & v_1 & v_3 & v_6
\end{bmatrix}
\begin{bmatrix}
1 & \alpha_{01} & \alpha_{03} & \alpha_{06} \\
1 & \alpha_{13} & \alpha_{16} \\
1 & \alpha_{36} \\
1
\end{bmatrix}^{-1}
\]

Let \( \mathcal{V} = \text{Span} \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\} \).
Suppose \( v_2^*, v_4^* \), and \( v_5^* \) are zero vectors.

\[
= \begin{bmatrix}
v_0^* & v_1^* & v_3^* & v_6^*
\end{bmatrix}
\]

Delete zero columns and the corresponding rows of the triangular matrix. Shows
\( \text{Span} \{v_0, v_1, v_2, v_3, v_4, v_5, v_6\} \subseteq \text{Span} \{v_0^*, v_1^*, v_3^*, v_6^*\} \)
so \( \{v_0^*, v_1^*, v_3^*, v_6^*\} \) is a basis for \( \mathcal{V} \)
Delete corresponding original columns \( v_2, v_4, v_5 \).
Resulting triangular matrix is invertible. Move it to other side.
Shows \( \text{Span} \{v_0^*, v_1^*, v_2^*, v_6^*\} \subseteq \text{Span} \{v_0, v_1, v_3, v_6\} \) so \( \{v_0, v_1, v_3, v_6\} \) is basis for \( \mathcal{V} \). QED
Roundoff error in computing a basis

In principle the following algorithm computes a basis for \( \text{Span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_n \} \):

```python
def find_basis([\mathbf{v}_1, \ldots, \mathbf{v}_n])
    Use orthogonalize to compute [\mathbf{v}_1^*, \ldots, \mathbf{v}_n^*]
    Return the list consisting of the nonzero vectors in this list.
```

However: the computer uses floating-point calculations. Due to round-off error, the vectors that are supposed to be zero won’t be exactly zero. Instead, consider a vector \( \mathbf{v} \) to be zero if \( \mathbf{v} \cdot \mathbf{v} \) is very small (e.g. smaller than \( 10^{-20} \)):

```python
def find_basis([\mathbf{v}_1, \ldots, \mathbf{v}_n])
    Use orthogonalize to compute [\mathbf{v}_1^*, \ldots, \mathbf{v}_n^*]
    Return the list consisting of vectors in this list that are nearly zero vectors.
```

Can use this procedure in turn to define rank(vlist) and is_independent(vlist).
Algorithm for finding basis for null space

Now let's find null space of matrix with columns $v_1, \ldots, v_n$.

Here's the matrix equation expressing original vectors in terms of starred vectors:

$$
\begin{bmatrix}
v_0 & v_1 & v_2 & \cdots & v_n
\end{bmatrix} =
\begin{bmatrix}
v_0^* & v_1^* & v_2^* & \cdots & v_n^*
\end{bmatrix}
\begin{bmatrix}
1 & \alpha_{01} & \alpha_{02} & \cdots & \alpha_{0n}
1 & \alpha_{12} & 1 & \cdots & \alpha_{1n}
\vdots & \vdots & \vdots & \ddots & \vdots
\alpha_{2n} & 1 & \cdots & 1
\end{bmatrix}

Can transform this to express starred vectors in terms of original vectors.

$$
\begin{bmatrix}
v_0 & v_1 & v_2 & \cdots & v_n
\end{bmatrix}
\begin{bmatrix}
1 & \alpha_{01} & \alpha_{02} & \cdots & \alpha_{0n}
1 & \alpha_{12} & 1 & \cdots & \alpha_{1n}
\vdots & \vdots & \vdots & \ddots & \vdots
\alpha_{2n} & 1 & \cdots & 1
\end{bmatrix}^{-1}
= 
\begin{bmatrix}
v_0^* & v_1^* & v_2^* & \cdots & v_n^*
\end{bmatrix}
Basis for null space

\[
\begin{bmatrix}
v_0 & v_1 & v_2 & v_3 & v_4 & v_5 & v_6
\end{bmatrix}
\begin{bmatrix}
1 & \alpha_{01} & \alpha_{02} & \alpha_{03} & \alpha_{04} & \alpha_{05} & \alpha_{06} \\
1 & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} & \alpha_{16} \\
1 & \alpha_{23} & \alpha_{24} & \alpha_{25} & \alpha_{26} \\
1 & \alpha_{34} & \alpha_{35} & \alpha_{36} \\
1 & \alpha_{45} & \alpha_{46} \\
1 & \alpha_{56}
\end{bmatrix}^{-1}
\]

\[
= \begin{bmatrix}
v_0^* & v_1^* & v_2^* & v_3^* & v_4^* & v_5^* & v_6^*
\end{bmatrix}
\]

Suppose \(v_2^*, v_4^*, \) and \(v_5^*\) are (approximately) zero vectors.

- Corresp. cols of inverse triang. matrix are vecs of null space of leftmost matrix.
- These columns are clearly linearly independent so they span a basis of dimension 3.
- Rank-Nullity Theorem shows that the null space has dimension 3 so these columns are a basis for null space.
Computing basis for null space

Write
\[
\begin{bmatrix}
v_0 & \cdots & v_n
\end{bmatrix}
\begin{bmatrix}
T
\end{bmatrix}
= 
\begin{bmatrix}
v_0^* & \cdots & v_n^*
\end{bmatrix}
\]

Then one basis for null space consists of columns of \( T^{-1} \) corresponding to zero vectors among \( v_0^*, \ldots, v_n^* \).

How to compute columns of inverse of \( T \)?

Use matrix-matrix equation
\[
\begin{bmatrix}
T
\end{bmatrix}
\begin{bmatrix}
T^{-1}
\end{bmatrix}
= 
\begin{bmatrix}
1 & \cdots & 1
\end{bmatrix}
\]
Column \( i \) of \( T^{-1} \) is a solution to matrix-vector equation \( Tx = e_i \), where \( e_i \) is standard basis vector \( i \)

\[
e_i = [0, \ldots, 0, 1, 0, \ldots, 0]
\]

How to solve matrix-vector equation? **backward substitution**
Orthogonal complement

Let $\mathcal{U}$ be a subspace of $\mathcal{W}$.

For each vector $\mathbf{b}$ in $\mathcal{W}$, we can write $\mathbf{b} = \mathbf{b}^{\parallel \mathcal{U}} + \mathbf{b}^{\perp \mathcal{U}}$ where

- $\mathbf{b}^{\parallel \mathcal{U}}$ is in $\mathcal{U}$, and
- $\mathbf{b}^{\perp \mathcal{U}}$ is orthogonal to every vector in $\mathcal{U}$.

Let $\mathcal{V}$ be the set $\{ \mathbf{b}^{\perp \mathcal{U}} : \mathbf{b} \in \mathcal{W} \}$.

**Definition:** We call $\mathcal{V}$ the orthogonal complement of $\mathcal{U}$ in $\mathcal{W}$.

**Easy observations:**

- Every vector in $\mathcal{V}$ is orthogonal to every vector in $\mathcal{U}$.
- Every vector $\mathbf{b}$ in $\mathcal{W}$ can be written as the sum of a vector in $\mathcal{U}$ and a vector in $\mathcal{V}$.

Maybe $\mathcal{U} \oplus \mathcal{V} = \mathcal{W}$? To show direct sum of $\mathcal{U}$ and $\mathcal{V}$ is defined, we need to show that the only vector that is in both $\mathcal{U}$ and $\mathcal{V}$ is the zero vector.

Any vector $\mathbf{w}$ in both $\mathcal{U}$ and $\mathcal{V}$ is orthogonal to itself. Thus $0 = \langle \mathbf{w}, \mathbf{w} \rangle = \| \mathbf{w} \|^2$.

By Property N2 of norms, that means $\mathbf{w} = \mathbf{0}$.

Therefore $\mathcal{U} \oplus \mathcal{V} = \mathcal{W}$. **Recall:** $\dim \mathcal{U} + \dim \mathcal{V} = \dim \mathcal{U} \oplus \mathcal{V}$.
Orthogonal complement: example

**Example:** Let \( \mathcal{U} = \text{Span}\ \{[1, 1, 0, 0], [0, 0, 1, 1]\} \). Let \( \mathcal{V} \) denote the orthogonal complement of \( \mathcal{U} \) in \( \mathbb{R}^4 \). What vectors form a basis for \( \mathcal{V} \)?

Every vector in \( \mathcal{U} \) has the form \([a, a, b, b]\).

Therefore any vector of the form \([c, -c, d, -d]\) is orthogonal to every vector in \( \mathcal{U} \).

Every vector in \( \text{Span}\ \{[1, -1, 0, 0], [0, 0, 1, -1]\} \) is orthogonal to every vector in \( \mathcal{U} \).

... so \( \text{Span}\ \{[1, -1, 0, 0], [0, 0, 1, -1]\} \) is a subspace of \( \mathcal{V} \), the orthogonal complement of \( \mathcal{U} \) in \( \mathbb{R}^4 \).

Is it the whole thing?

\( \mathcal{U} \oplus \mathcal{V} = \mathbb{R}^4 \) so \( \dim \mathcal{U} + \dim \mathcal{V} = 4 \).

\( \{[1, 1, 0, 0], [0, 0, 1, 1]\} \) is linearly independent so \( \dim \mathcal{U} = 2 \) ... so \( \dim \mathcal{V} = 2 \)

\( \{[1, -1, 0, 0], [0, 0, 1, -1]\} \) is linearly independent
so \( \dim \text{Span}\ \{[1, -1, 0, 0], [0, 0, 1, -1]\} \) is also 2... so \( \text{Span}\ \{[1, -1, 0, 0], [0, 0, 1, -1]\} = \mathcal{V} \).
Orthogonal complement: example

Example: Find a basis for the null space of $A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 5 & 1 & 2 \\ 0 & 2 & 5 & 6 \end{bmatrix}$

By the dot-product definition of matrix-vector multiplication, a vector $v$ is in the null space of $A$ if the dot-product of each row of $A$ with $v$ is zero.

Thus the null space of $A$ equals the orthogonal complement of Row $A$ in $\mathbb{R}^4$.

Since the three rows of $A$ are linearly independent, we know $\dim \text{Row } A = 3$...

so the dimension of the orthogonal complement of Row $A$ in $\mathbb{R}^4$ is $4 - 3 = 1$....

The vector $[1, \frac{1}{10}, \frac{13}{20}, \frac{-23}{40}]$ has a dot-product of zero with every row of $A$...

so this vector forms a basis for the orthogonal complement.

and thus a basis for the null space of $A$. 
Using orthogonalization to find intersection of geometric objects

**Example:** Find the intersection of

- the plane spanned by \([1, 2, -2]\) and \([0, 1, 1]\)
- the plane spanned by \([1, 0, 0]\) and \([0, 1, -1]\)

The orthogonal complement in \(\mathbb{R}^3\) of the first plane is \(\text{Span}\ \{(4, -1, 1)\}\).
Therefore first plane is \(\{[x, y, z] \in \mathbb{R}^3 : [4, -1, 1] \cdot [x, y, z] = 0\}\)

The orthogonal complement in \(\mathbb{R}^3\) of the second plane is \(\text{Span}\ \{(0, 1, 1)\}\).
Therefore second plane is \(\{[x, y, z] \in \mathbb{R}^3 : [0, 1, 1] \cdot [x, y, z] = 0\}\)

The intersection of these two sets is the set
\(\{[x, y, z] \in \mathbb{R}^3 : [4, -1, 1] \cdot [x, y, z] = 0\ \text{and} \ [0, 1, 1] \cdot [x, y, z] = 0\}\)

How to find a basis for this solution set? We saw (in Lecture of 10/30) that is is just a basis for the null space of \(A = \begin{bmatrix} 4 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\)

The null space of \(A\) is the orthogonal complement of \(\text{Span}\ \{(4, -1, 1), [0, 1, 1]\}\) in \(\mathbb{R}^3\)...

which is \(\text{Span}\ \{[1, 2, -2]\}\)