Activity: Derive a matrix from input-output pairs

The $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ satisfies the following equations:

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 35 \\ 35 \end{bmatrix}$

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

Calculate the entries of the matrix.
Wiimote whiteboard

For location of infrared point, wiimote provides coordinate representation in terms of its camera basis.

To use as a mouse, need to find corresponding location on screen (coordinate representation in terms of screen basis)

How to transform from one coordinate representation to the other?

Can do this using a matrix $H$.

The challenge is to calculate the matrix $H$.

Can do this if you know the camera coordinate representation of four points whose screen coordinate representations are known.

You’ll do exactly the same computation but for a slightly different problem....
Removing perspective

Given an image of a whiteboard, taken from an angle...

synthesize an image from straight ahead with no perspective
Camera coordinate system

We use same camera-oriented basis $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$:

- The origin is the camera center.
- The first vector $\mathbf{a}_1$ goes horizontally from the top-left corner of a sensor element to the top-right corner.
- The second vector $\mathbf{a}_2$ goes vertically from the top-left corner of the sensor array to the bottom-left corner.
- The third vector $\mathbf{a}_3$ goes from the origin (the camera center) to the top-left corner of sensor element $(0,0)$. 
Converting from one basis to another

In addition, we define a whiteboard basis $\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$

- The origin is the camera center.
- The first vector $\mathbf{c}_1$ goes horizontally from the top-left corner of whiteboard to top-right corner.
- The second vector $\mathbf{c}_2$ goes vertically from the top-left corner of whiteboard to the bottom-left corner.
- The third vector $\mathbf{c}_3$ goes from the origin (the camera center) to the top-left corner of whiteboard.
Converting between different basis representations

Start with a point \( \mathbf{p} \) written in terms of in camera coordinates

\[
\mathbf{p} = \begin{bmatrix}
a_1 & a_2 & a_3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\]

We write the same point \( \mathbf{p} \) in the whiteboard coordinate system as

\[
\mathbf{p} = \begin{bmatrix}
c_1 & c_2 & c_3
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix}
\]

Combining the two equations, we obtain

\[
\begin{bmatrix}
a_1 & a_2 & a_3
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
c_1 & c_2 & c_3
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix}
\]
Let $A$ and $C$ be the two matrices. As before, $C$ has an inverse $C^{-1}$. Multiplying equation on the left by $C^{-1}$, we obtain

$$
\begin{bmatrix}
C^{-1} \\
A
\end{bmatrix}

\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}

= 

\begin{bmatrix}
C^{-1} \\
C
\end{bmatrix}

\begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix}

Since $C^{-1}$ and $C$ cancel out,

$$
\begin{bmatrix}
C^{-1} \\
A
\end{bmatrix}

\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}

= 

\begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix}

We have shown that there is a matrix $H$ (namely $H = C^{-1}A$) such that

$$
\begin{bmatrix}
H
\end{bmatrix}

\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}

= 

\begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix}$$
From pixel coordinates to whiteboard coordinates

\[(x_1, x_2)\]
\[\downarrow\]
\[(x_1, x_2, 1)\]
represent point in image plane in terms of camera basis
change to representation in terms of whiteboard basis

\[(y_1, y_2, y_3) = H \times (x_1, x_2, 1)\]
move to corresponding point in whiteboard plane

\[\downarrow\]
\[\left(\frac{y_1}{y_3}, \frac{y_2}{y_3}, \frac{y_3}{y_3}\right)\]
get coordinates within whiteboard

\[\downarrow\]
\[\left(\frac{y_1}{y_3}, \frac{y_2}{y_3}\right)\]
Activity: Derive a matrix (up to a scale factor)

The $2 \times 2$ matrix $A$ satisfies the following equations:

1. \[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\begin{bmatrix}
  2 \\
  3
\end{bmatrix} = 
\begin{bmatrix}
  y_1 \\
  y_2
\end{bmatrix}
\]
   where \[
\begin{bmatrix}
  y_1/y_2 \\
  y_2/y_2
\end{bmatrix} = 
\begin{bmatrix}
  2 \\
  1
\end{bmatrix}
\]

2. \[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\begin{bmatrix}
  4 \\
  1
\end{bmatrix} = 
\begin{bmatrix}
  y_3 \\
  y_4
\end{bmatrix}
\]
   where \[
\begin{bmatrix}
  y_3/y_4 \\
  y_4/y_4
\end{bmatrix} = 
\begin{bmatrix}
  1 \\
  1
\end{bmatrix}
\]

3. \[
\begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\begin{bmatrix}
  -1 \\
  1
\end{bmatrix} = 
\begin{bmatrix}
  y_5 \\
  y_6
\end{bmatrix}
\]
   where \[
\begin{bmatrix}
  y_5/y_6 \\
  y_6/y_6
\end{bmatrix} = 
\begin{bmatrix}
  0 \\
  1
\end{bmatrix}
\]

Calculate the entries of the matrix *up to a scale factor*. That is, you are allowed to choose an arbitrary scale for the matrix. If your matrix is a scalar multiple of the true matrix, your answer is considered correct.
How to almost compute $H$

Write $H = \begin{bmatrix} h_{y_1,x_1} & h_{y_1,x_2} & h_{y_1,x_3} \\ h_{y_2,x_1} & h_{y_2,x_2} & h_{y_2,x_3} \\ h_{y_3,x_1} & h_{y_3,x_2} & h_{y_3,x_3} \end{bmatrix}$

The $h_{ij}$’s are the unknowns.

To derive equations, let $\mathbf{p}$ be some point on the whiteboard, and let $\mathbf{q}$ be the corresponding point on the image plane. Let $(x_1, x_2, 1)$ be the camera coordinates of $\mathbf{q}$, and let $(y_1, y_2, y_3)$ be the whiteboard coordinates of $\mathbf{q}$. We have

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} h_{y_1,x_1} & h_{y_1,x_2} & h_{y_1,x_3} \\ h_{y_2,x_1} & h_{y_2,x_2} & h_{y_2,x_3} \\ h_{y_3,x_1} & h_{y_3,x_2} & h_{y_3,x_3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$$

Multiplying out, we obtain

$$y_1 = h_{y_1,x_1} x_1 + h_{y_1,x_2} x_2 + h_{y_1,x_3}$$
$$y_2 = h_{y_2,x_1} x_1 + h_{y_2,x_2} x_2 + h_{y_2,x_3}$$
$$y_3 = h_{y_3,x_1} x_1 + h_{y_3,x_2} x_2 + h_{y_3,x_3}$$
Almost computing $H$

\[
\begin{align*}
y_1 &= h_{y_1,x_1}x_1 + h_{y_1,x_2}x_2 + h_{y_1,x_3} \\
y_2 &= h_{y_2,x_1}x_1 + h_{y_2,x_2}x_2 + h_{y_2,x_3} \\
y_3 &= h_{y_3,x_1}x_1 + h_{y_3,x_2}x_2 + h_{y_3,x_3}
\end{align*}
\]

Whiteboard coordinates of the original point $p$ are $(y_1/y_3, y_2/y_3, 1)$. Define

\[
\begin{align*}
w_1 &= y_1/y_3 \\
w_2 &= y_2/y_3
\end{align*}
\]

so the whiteboard coordinates of $p$ are $(w_1, w_2, 1)$. Multiplying through by $y_3$, we obtain

\[
\begin{align*}
w_1y_3 &= y_1 \\
w_2y_3 &= y_2
\end{align*}
\]

Substituting our expressions for $y_1, y_2, y_3$, we obtain

\[
\begin{align*}
w_1(h_{y_3,x_1}x_1 + h_{y_3,x_2}x_2 + h_{y_3,x_3}) &= h_{y_1,x_1}x_1 + h_{y_1,x_2}x_2 + h_{y_1,x_3} \\
w_2(h_{y_3,x_1}x_1 + h_{y_3,x_2}x_2 + h_{y_3,x_3}) &= h_{y_2,x_1}x_1 + h_{y_2,x_2}x_2 + h_{y_2,x_3}
\end{align*}
\]
\[
\begin{align*}
  w_1(h_{y_3,x_1}x_1 + h_{y_3,x_2}x_2 + h_{y_3,x_3}) &= h_{y_1,x_1}x_1 + h_{y_1,x_2}x_2 + h_{y_1,x_3} \\
  w_2(h_{y_3,x_1}x_1 + h_{y_3,x_2}x_2 + h_{y_3,x_3}) &= h_{y_2,x_1}x_1 + h_{y_2,x_2}x_2 + h_{y_2,x_3}
\end{align*}
\]

Multiplying through and moving everything to the same side, we obtain

\[
\begin{align*}
  (w_1x_1)h_{y_3,x_1} + (w_1x_2)h_{y_3,x_2} + w_1h_{y_3,x_3} - x_1h_{y_1,x_1} - x_2h_{y_1,x_2} - 1h_{y_1,x_3} &= 0 \\
  (w_2x_1)h_{y_3,x_1} + (w_2x_2)h_{y_3,x_2} + w_2h_{y_3,x_3} - x_1h_{y_2,x_1} - x_2h_{y_2,x_2} - 1h_{y_2,x_3} &= 0
\end{align*}
\]

Thus we get two linear equations in the unknowns. The coefficients are expressed in terms of \( x_1, x_2, w_1, w_2 \).

For four points, get eight equations. Need one more...
We can't pin down $H$ precisely.

This corresponds to the fact that we cannot recover the scale of the picture (a tiny building that is nearby looks just like a huge building that is far away).

Fortunately, we don’t need the true $H$.

As long as the $H$ we compute is a scalar multiple of the true $H$, things will work out.

To arbitrarily select a scale, we add the equation $h_{y_1,x_1} = 1$. 
Once you know $H$

1. For each point $q$ in the representation of the image, we have the camera coordinates $(x_1, x_2, 1)$ of $q$. We multiply by $H$ to obtain the whiteboard coordinates $(y_1, y_2, y_3)$ of the same point $q$.

2. Recall the situation as viewed from above:

The whiteboard coordinates of the corresponding point $p$ on the whiteboard are $(y_1 / y_3, y_2 / y_3, 1)$. Use this formula to compute these coordinates.

3. Display the updated points with the same color matrix.
Quiz

Draw diagrams showing

- one way in which a subset of a cartesian product $A \times B$ can fail to be a function from $A$ to $B$, and
- a second way;

- one way in which a function from $A$ to $B$ can fail to be invertible,
- a second way;
Simplified Exchange Lemma

We need a tool to iteratively transform one set of generators into another.

- You have a set $S$ of vectors.
- You have a vector $z$ you want to inject into $S$.
- You want to maintain same size so must eject a vector from $S$.
- You want the span to not change.

Exchange Lemma tells you how to choose vector to eject.

**Simplified Exchange Lemma:**

- Suppose $S$ is a set of vectors.
- Suppose $z$ is a nonzero vector in $\text{Span } S$.
- Then there is a vector $w$ in $S$ such that

$$\text{Span } (S \cup \{z\} - \{w\}) = \text{Span } S$$
Simplified Exchange Lemma proof

**Simplified Exchange Lemma:** Suppose $S$ is a set of vectors, and $z$ is a nonzero vector in $\text{Span } S$. Then there is a vector $w$ in $S$ such that $\text{Span } (S \cup \{z\} - \{w\}) = \text{Span } S$.

**Proof:** Let $S = \{v_1, \ldots, v_n\}$. Since $z$ is in $\text{Span } S$, can write

$$z = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

By Superfluous-Vector Lemma, $\text{Span } (S \cup \{z\}) = \text{Span } S$.

Since $z$ is nonzero, at least one of the coefficients is nonzero, say $\alpha_i$.

Rewrite as

$$z - \alpha_1 v_1 - \cdots - \alpha_{i-1} v_{i-1} - \alpha_{i+1} v_{i+1} - \cdots - \alpha_n v_n = \alpha_i v_i$$

Divide through by $\alpha_i$:

$$\frac{1}{\alpha_i}z - \left(\frac{\alpha_1}{\alpha_i}\right) v_1 - \cdots - \left(\frac{\alpha_{i-1}}{\alpha_i}\right) v_{i-1} - \left(\frac{\alpha_{i+1}}{\alpha_i}\right) v_{i+1} - \cdots - \left(\frac{\alpha_n}{\alpha_i}\right) v_n = v_i$$

By Superfluous-Vector Lemma, $\text{Span } (S \cup \{z\}) = \text{Span } (S \cup \{z\} - \{w\})$. QED
**Simplified Exchange Lemma:** Suppose $S$ is a set of vectors, and $z$ is a nonzero vector in $\text{Span } S$. Then there is a vector $w$ in $S$ such that $\text{Span } (S \cup \{z\} - \{w\}) = \text{Span } S$.

Simplified Exchange Lemma helps in transforming one generating set into another...

Trying to put squares in—when you put in one square, you might end up taking out a previously inserted square.

Need a way to protect some elements from being taken out.
**Simplified Exchange Lemma:** Suppose $S$ is a set of vectors, and $z$ is a nonzero vector in $\text{Span } S$. Then there is a vector $w$ in $S$ such that $\text{Span } (S \cup \{z\} - \{w\}) = \text{Span } S$.

Simplified Exchange Lemma helps in transforming one generating set into another...

inject

Trying to put squares in—when you put in one square, you might end up taking out a previously inserted square

Need a way to protect some elements from being taken out.
Simplified Exchange Lemma: Suppose $S$ is a set of vectors, and $z$ is a nonzero vector in $\text{Span} \ S$. Then there is a vector $w$ in $S$ such that

$$\text{Span} \ (S \cup \{z\} - \{w\}) = \text{Span} \ S.$$
**Simplified Exchange Lemma:** Suppose $S$ is a set of vectors, and $z$ is a nonzero vector in $\text{Span } S$. Then there is a vector $w$ in $S$ such that $\text{Span } (S \cup \{z\} - \{w\}) = \text{Span } S$.

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Simplified Exchange Lemma: Suppose $S$ is a set of vectors, and $z$ is a nonzero vector in $\text{Span } S$. Then there is a vector $w$ in $S$ such that $\text{Span } (S \cup \{z\} - \{w\}) = \text{Span } S$.

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Need a way to protect some elements from being taken out.
Exchange Lemma

**Simplified Exchange Lemma:** Suppose $S$ is a set of vectors, and $z$ is a nonzero vector in $\text{Span } S$. Then there is a vector $w$ in $S$ such that $\text{Span } (S \cup \{z\} - \{w\}) = \text{Span } S$.

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Simplified Exchange Lemma: Suppose \( S \) is a set of vectors, and \( z \) is a nonzero vector in \( \text{Span } S \). Then there is a vector \( w \) in \( S \) such that
\[
\text{Span } (S \cup \{z\} - \{w\}) = \text{Span } S.
\]

Simplified Exchange Lemma helps in transforming one generating set into another...

Trying to put squares in—when you put in one square, you might end up taking out a previously inserted square.

Need a way to protect some elements from being taken out.
Simplified Exchange Lemma: Suppose $S$ is a set of vectors, and $z$ is a nonzero vector in $\text{Span } S$. Then there is a vector $w$ in $S$ such that

$$\text{Span } (S \cup \{z\} - \{w\}) = \text{Span } S.$$
**Simplified Exchange Lemma:** Suppose $S$ is a set of vectors, and $z$ is a nonzero vector in $\text{Span } S$. Then there is a vector $w$ in $S$ such that $\text{Span } (S \cup \{z\} - \{w\}) = \text{Span } S$.

Simplified Exchange Lemma helps in transforming one generating set into another...

Trying to put squares in—when you put in one square, you might end up taking out a previously inserted square.

Need a way to protect some elements from being taken out.
Simplified Exchange Lemma: Suppose $S$ is a set of vectors, and $z$ is a nonzero vector in $\text{Span } S$. Then there is a vector $w$ in $S$ such that $\text{Span } (S \cup \{z\} \setminus \{w\}) = \text{Span } S$.

Simplified Exchange Lemma helps in transforming one generating set into another...

Trying to put squares in—when you put in one square, you might end up taking out a previously inserted square

Need a way to protect some elements from being taken out.
**Simplified Exchange Lemma:** Suppose $S$ is a set of vectors, and $z$ is a nonzero vector in $\text{Span } S$. Then there is a vector $w$ in $S$ such that $\text{Span } (S \cup \{z\} - \{w\}) = \text{Span } S$.

Need to enhance this lemma. Set of *protected* elements is $A$:

**Exchange Lemma:**

- Suppose $S$ is a set of vectors and $A$ is a subset of $S$.
- Suppose $z$ is a vector in $\text{Span } S$ such that $A \cup \{z\}$ is linearly independent.
- Then there is a vector $w \in S - A$ such that $\text{Span } S = \text{Span } (S \cup \{z\} - \{w\})$

Now, not enough that $z$ be nonzero—need $A$ to be linearly independent.
Exchange Lemma proof

**Exchange Lemma:** Suppose $S$ is a set of vectors and $A$ is a subset of $S$. Suppose $z$ is a vector in $\text{Span } S$ such that $A \cup \{z\}$ is linearly independent. Then there is a vector $w \in S - A$ such that $\text{Span } S = \text{Span } (S \cup \{z\} - \{w\})$

**Proof:** Let $S = \{v_1, \ldots, v_k, w_1, \ldots, w_\ell\}$ and $A = \{v_1, \ldots, v_k\}$. Since $z$ is in $\text{Span } S$, can write

$$z = \alpha_1 v_1 + \cdots + \alpha_k v_k + \beta_1 w_1 + \cdots + \beta_\ell w_\ell$$

By Superfluous-Vector Lemma, $\text{Span } (S \cup \{z\}) = \text{Span } S$. If coefficients $\beta_1, \ldots, \beta_\ell$ were all zero then we would have $z = \alpha_1 v_1 + \cdots + \alpha_k v_k$, contradicting the linear independence of $A \cup \{z\}$. Thus one of the coefficients $\beta_1, \ldots, \beta_\ell$ must be nonzero... say $\beta_1$. Rewrite as

$$z - \alpha_1 v_1 - \cdots - \alpha_k v_k - \beta_2 w_2 - \cdots - \beta_\ell w_\ell = \beta_1 w_1$$

Divide through by $\beta_1$:

$$\frac{1}{\beta_1}z - \frac{\alpha_1}{\beta_1} v_1 - \cdots - \frac{\alpha_k}{\beta_1} v_k - \frac{\beta_2}{\beta_1} w_2 - \cdots - \frac{\beta_\ell}{\beta_1} w_\ell = w_1$$

By Superfluous-Vector Lemma, $\text{Span } (S \cup \{z\}) = \text{Span } (S \cup \{z\} - \{w_1\})$. QED