Activity

Suppose you have available a procedure `is_independent(L)`, which takes a list `L` of Vecs and returns True or False depending on whether the vectors are independent or not.

Write a procedure

```python
def has_many_solutions(a_list, b_list, u):
```

with the following spec:

- **input:** list `[a_1, \ldots, a_n]` of Vecs, list `[\beta_1, \ldots, \beta_n]` of scalars, Vec `u` that is a solution to the linear system
  \[
  a_1 \cdot x = \beta_1 \\
  \vdots \\
  a_n \cdot x = \beta_n
  \]

- **output:** True if there are solutions other than `u` to the linear system.
So far we've done \( \text{paths} = \text{spanning} \) and \( \text{cycles} = \text{linearly dependent} \) over \( \text{GF}(2) \). How would you achieve the same over \( \mathbb{R} \)?
Properties of linear (in)dependence

**Linear-Dependence Lemma** Let \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) be vectors. A vector \( \mathbf{v}_i \) is in the span of the other vectors if and only if the zero vector can be written as a linear combination of \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) in which the coefficient of \( \mathbf{v}_i \) is nonzero.

**Contrapositive:**

\( \mathbf{v}_i \) is *not* in the space of the other vectors if and only if for any linear combination equaling the zero vector

\[
0 = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_i \mathbf{v}_i + \cdots + \alpha_n \mathbf{v}_n
\]

it must be that the coefficient \( \alpha_i \) *is* zero.
Analyzing the Grow algorithm

```
def GROW(\mathcal{V})
    S = \emptyset
    repeat while possible:
        find a vector \textbf{v} in \mathcal{V} that is not in Span \ S, and put it in \ S.
```

**Grow-Algorithm Corollary:** The vectors obtained by the Grow algorithm are linearly independent.

In graphs, this means that the solution obtained by the Grow algorithm has no cycles (is a forest).
Analyzing the Grow algorithm

**Grow-Algorithm Corollary:** The vectors obtained by the Grow algorithm are linearly independent.

**Proof:** For \( n = 1, 2, \ldots \), let \( \mathbf{v}_n \) be the vector added to \( S \) in the \( n^{th} \) iteration of the Grow algorithm. We show by induction that \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \) are linearly independent.

For \( n = 0 \), there are no vectors, so the claim is trivially true.

Assume the claim is true for \( n = k - 1 \). We prove it for \( n = k \).

The vector \( \mathbf{v}_k \) added to \( S \) in the \( k^{th} \) iteration is not in the span of \( \mathbf{v}_1, \ldots, \mathbf{v}_{k-1} \).

Therefore, by the Linear-Dependence Lemma, for any coefficients \( \alpha_1, \ldots, \alpha_k \) such that

\[
0 = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_{k-1} \mathbf{v}_{k-1} + \alpha_k \mathbf{v}_k
\]

it must be that \( \alpha_k \) equals zero. We may therefore write

\[
0 = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_{k-1} \mathbf{v}_{k-1}
\]

By claim for \( n = k - 1 \), \( \mathbf{v}_1, \ldots, \mathbf{v}_{k-1} \) are linearly independent, so \( \alpha_1 = \cdots = \alpha_{k-1} = 0 \).

The linear combination of \( \mathbf{v}_1, \ldots, \mathbf{v}_k \) is trivial. We have proved that \( \mathbf{v}_1, \ldots, \mathbf{v}_k \) are linearly independent. This proves the claim for \( n = k \).

QED
Analyzing the Shrink algorithm

def Shrink(\mathcal{V})
    \[ S = \text{some finite set of vectors that spans } \mathcal{V} \]
    repeat while possible:
        find a vector \( \mathbf{v} \) in \( S \) such that Span \( (S - \{ \mathbf{v} \}) = \mathcal{V} \), and remove \( \mathbf{v} \) from \( S \).

**Shrink-Algorithm Corollary:** The vectors obtained by the Shrink algorithm are linearly independent.

In graphs, this means that the Shrink algorithm outputs a solution that is a forest.

*Recall:*

**Superfluous-Vector Lemma** For any set \( S \) and any vector \( \mathbf{v} \in S \), if \( \mathbf{v} \) can be written as a linear combination of the other vectors in \( S \) then 
\[ \text{Span} \ (S - \{ \mathbf{v} \}) = \text{Span} \ S \]
Analyzing the Shrink algorithm

**Shrink-Algorithm Corollary:** The vectors obtained by the Shrink algorithm are linearly independent.

**Proof:** Let $S = \{v_1, \ldots, v_n\}$ be the set of vectors obtained by the Shrink algorithm. Assume for a contradiction that the vectors are linearly dependent.

Then $0$ can be written as a nontrivial linear combination

$$0 = \alpha_1 v_1 + \cdots + \alpha_n v_n$$

where at least one of the coefficients is nonzero.

Let $\alpha_i$ be one of the nonzero coefficients.

By the Linear-Dependence Lemma, $v_i$ can be written as a linear combination of the other vectors.

Hence by the Superfluous-Vector Lemma, $\text{Span } (S - \{v_i\}) = \text{Span } S$, so the Shrink algorithm should have removed $v_i$.  

QED
Basis

If they successfully finish, the Grow algorithm and the Shrink algorithm each find a set of vectors spanning the vector space $V$. In each case, the set of vectors found is linearly independent.

**Definition:** Let $V$ be a vector space. A *basis* for $V$ is a linearly independent set of generators for $V$.

Thus a set $S$ of vectors of $V$ is a *basis* for $V$ if $S$ satisfies two properties:

**Property B1** *(Spanning)* $\text{Span } S = V$, and

**Property B2** *(Independent)* $S$ is linearly independent.

*Most important definition in linear algebra.*
A set $S$ of vectors of $V$ is a *basis* for $V$ if $S$ satisfies two properties:

**Property B1 (Spanning)** $\text{Span } S = V$, and

**Property B2 (Independent)** $S$ is linearly independent.

**Example:** Let $V = \text{Span } \{[1, 0, 2, 0], [0, -1, 0, -2], [2, 2, 4, 4]\}$. Is $\{[1, 0, 2, 0], [0, -1, 0, -2], [2, 2, 4, 4]\}$ a basis for $V$?

The set *is* spanning but is *not* independent

$$1[1, 0, 2, 0] - 1[0, -1, 0, -2] - \frac{1}{2}[2, 2, 4, 4] = 0$$

so not a basis

However, $\{[1, 0, 2, 0], [0, -1, 0, -2]\}$ *is* a basis:

- Obvious that these vectors are independent because each has a nonzero entry where the other has a zero.
- To show
  $$\text{Span } \{[1, 0, 2, 0], [0, -1, 0, -2]\} = \text{Span } \{[1, 0, 2, 0], [0, -1, 0, -2], [2, 2, 4, 4]\},$$
  can use Superfluous-Vector Lemma:
  $$[2, 2, 4, 4] = 2[1, 0, 2, 0] - 2[0, -1, 0, -2]$$
**Example:** A simple basis for $\mathbb{R}^3$: the standard generators

$e_1 = [1, 0, 0], e_2 = [0, 1, 0], e_3 = [0, 0, 1]$.

- **Spanning:** For any vector $[x, y, z] \in \mathbb{R}^3$,

  \[
  [x, y, z] = x [1, 0, 0] + y [0, 1, 0] + z [0, 0, 1]
  \]

- **Independent:** Suppose

  \[
  0 = \alpha_1 [1, 0, 0] + \alpha_2 [0, 1, 0] + \alpha_3 [0, 0, 1] = [\alpha_1, \alpha_2, \alpha_3]
  \]

  Then $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Instead of “standard generators”, we call them *standard basis vectors*. We refer to $\{[1, 0, 0], [0, 1, 0], [0, 0, 1]\}$ as *standard basis* for $\mathbb{R}^3$.

In general the standard generators are usually called *standard basis vectors*. 
Basis: Examples

Example: Another basis for $\mathbb{R}^3$: $[1, 1, 1], [1, 1, 0], [0, 1, 1]$

- **Spanning**: Can write standard generators in terms of these vectors:

  $[1, 0, 0] = [1, 1, 1] - [0, 1, 1]$

  $[0, 1, 0] = [1, 1, 0] + [0, 1, 1] - [1, 1, 1]$

  $[0, 0, 1] = [1, 1, 1] - [1, 1, 0]$

Since $e_1, e_2, e_3$ can be written in terms of these new vectors, every vector in $\text{Span} \{ e_1, e_2, e_3 \}$ is in span of new vectors. Thus $\mathbb{R}^3$ equals span of new vectors.

- **Linearly independent**: Write zero vector as linear combination:

  $0 = x [1, 1, 1] + y [1, 1, 0] + z [0, 1, 1] = [x + y, x + y + z, x + z]$

Looking at each entry, we get

- $0 = x + y$
- $0 = x + y + z$
- $0 = x + z$

Plug $x + y = 0$ into second equation to get $0 = z$.

Plug $z = 0$ into third equation to get $x = 0$.

Plug $x = 0$ into first equation to get $y = 0$.

Thus the linear combination is trivial.
One kind of basis in a graph $G$: a set $S$ of edges forming a spanning forest.

- **Spanning**: for each edge $xy$ in $G$, there is an $x$-to-$y$ path consisting of edges of $S$.
- **Independent**: no cycle consisting of edges of $S$
Towards showing that every vector space has a basis

We would like to prove that every vector space $\mathcal{V}$ has a basis.

The Grow algorithm and the Shrink algorithm each provides a way to prove this, but we are not there yet:

- The Grow-Algorithm Corollary implies that, if the Grow algorithm terminates, the set of vectors it has selected is a basis for the vector space $\mathcal{V}$. However, we have not yet shown that it always terminates!

- The Shrink-Algorithm Corollary implies that, if we can run the Shrink algorithm starting with a finite set of vectors that spans $\mathcal{V}$, upon termination it will have selected a basis for $\mathcal{V}$. However, we have not yet shown that every vector space $\mathcal{V}$ is spanned by some finite set of vectors!.
Computational problems involving finding a basis

Two natural ways to specify a vector space $\mathcal{V}$:

1. Specifying generators for $\mathcal{V}$.
2. Specifying a homogeneous linear system whose solution set is $\mathcal{V}$.

Two Fundamental Computational Problems:

**Computational Problem:** Finding a basis of the vector space spanned by given vectors

- **input:** a list $[\mathbf{v}_1, \ldots, \mathbf{v}_n]$ of vectors
- **output:** a list of vectors that form a basis for $\text{Span} \{ \mathbf{v}_1, \ldots, \mathbf{v}_n \}$.

**Computational Problem:** Finding a basis of the solution set of a homogeneous linear system

- **input:** a list $[\mathbf{a}_1, \ldots, \mathbf{a}_n]$ of vectors
- **output:** a list of vectors that form a basis for the set of solutions to the system $\mathbf{a}_1 \cdot \mathbf{x} = 0, \ldots, \mathbf{a}_n \cdot \mathbf{x} = 0$