Quiz

What are the requirements for a set $V$ to be a subspace of $\mathbb{F}^D$?

Give an example (by specifying $\mathbb{F}$, $D$, and $V$ such that $V$ is a subspace of $\mathbb{F}^D$) in which $V$ is infinite.

Give an example (by specifying $\mathbb{F}$, $D$, and $V$ such that $V$ is a subspace of $\mathbb{F}^D$) in which $V$ is finite.

What does the notion of $\text{Span} \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ have to do with the notion of subspace?

What does the notion of $\{\mathbf{x} : \mathbf{a}_1 \cdot \mathbf{x} = 0, \mathbf{a}_2 \cdot \mathbf{x} = 0, \mathbf{a}_3 \cdot \mathbf{x} = 0\}$ have to do with the notion of subspace?
Ungraded part of quiz

- What is an abstract vector space?

- Prove that
  - \( \text{Span} \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \), or
  - \( \{\mathbf{x} : a_1 \cdot \mathbf{x} = 0, a_2 \cdot \mathbf{x} = 0, a_3 \cdot \mathbf{x} = 0\} \)

  is a vector space. (Choose one.)
Geometric objects that exclude the origin

How to represent a line that does \textit{not} contain the origin?

Start with a line that \textit{does} contain the origin.

We know that points of such a line form a vector space $\mathcal{V}$.

Translate the line by adding a vector $\mathbf{c}$ to every vector in $\mathcal{V}$:

$$\{ \mathbf{c} + \mathbf{v} : \mathbf{v} \in \mathcal{V} \}$$

(abbreviated $\mathbf{c} + \mathcal{V}$)

Result is line through $\mathbf{c}$ instead of through origin.
Geometric objects that exclude the origin

How to represent a plane that does not contain the origin?

Start with a plane that does contain the origin. We know that points of such a plane form a vector space \( V \).

Translate it by adding a vector \( c \) to every vector in \( V \)

\[ \{ c + v : v \in V \} \]

(abbreviated \( c + V \))

Result is plane containing \( c \).
Affine space

**Definition:** If $c$ is a vector and $\mathcal{V}$ is a vector space then $c + \mathcal{V}$ is called an *affine space*.

**Examples:** A plane or a line not necessarily containing the origin.
Affine space and affine combination

**Example:** The plane containing $\mathbf{u}_1 = [3, 0, 0]$, $\mathbf{u}_2 = [-3, 1, -1]$, and $\mathbf{u}_3 = [1, -1, 1]$.

Want to express this plane as $\mathbf{u}_1 + \mathcal{V}$ where $\mathcal{V}$ is the span of two vectors (a plane containing the origin)

Let $\mathcal{V} = \text{Span} \left\{ \mathbf{a}, \mathbf{b} \right\}$ where

$$\mathbf{a} = \mathbf{u}_2 - \mathbf{u}_1 \quad \text{and} \quad \mathbf{b} = \mathbf{u}_3 - \mathbf{u}_1$$

Since $\mathbf{u}_1 + \mathcal{V}$ is a translation of a plane, it is also a plane.

- Span $\{\mathbf{a}, \mathbf{b}\}$ contains $\mathbf{0}$, so $\mathbf{u}_1 + \text{Span} \left\{ \mathbf{a}, \mathbf{b} \right\}$ contains $\mathbf{u}_1$.
- Span $\{\mathbf{a}, \mathbf{b}\}$ contains $\mathbf{u}_2 - \mathbf{u}_1$ so $\mathbf{u}_1 + \text{Span} \left\{ \mathbf{a}, \mathbf{b} \right\}$ contains $\mathbf{u}_2$.
- Span $\{\mathbf{a}, \mathbf{b}\}$ contains $\mathbf{u}_3 - \mathbf{u}_1$ so $\mathbf{u}_1 + \text{Span} \left\{ \mathbf{a}, \mathbf{b} \right\}$ contains $\mathbf{u}_3$.

Thus the plane $\mathbf{u}_1 + \text{Span} \left\{ \mathbf{a}, \mathbf{b} \right\}$ contains $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

Only one plane contains those three points, so this is that one.
Affine space and affine combination

Example: The plane containing \(\mathbf{u}_1 = [3, 0, 0]\), \(\mathbf{u}_2 = [-3, 1, -1]\), and \(\mathbf{u}_1 = [1, -1, 1]\):

\[
\mathbf{u}_1 + \text{Span} \ \{\mathbf{u}_2 - \mathbf{u}_1, \mathbf{u}_3 - \mathbf{u}_1\}
\]

Cleaner way to write it?

\[
\mathbf{u}_1 + \text{Span} \ \{\mathbf{u}_2 - \mathbf{u}_1, \mathbf{u}_3 - \mathbf{u}_1\} = \{\mathbf{u}_1 + \alpha (\mathbf{u}_2 - \mathbf{u}_1) + \beta (\mathbf{u}_3 - \mathbf{u}_1) : \alpha, \beta \in \mathbb{R}\}
\]

\[
= \{\mathbf{u}_1 + \alpha \mathbf{u}_2 - \alpha \mathbf{u}_1 + \beta \mathbf{u}_3 - \beta \mathbf{u}_1 : \alpha, \beta \in \mathbb{R}\}
\]

\[
= \{(1 - \alpha - \beta) \mathbf{u}_1 + \alpha \mathbf{u}_2 + \beta \mathbf{u}_3 : \alpha, \beta \in \mathbb{R}\}
\]

\[
= \{\gamma \mathbf{u}_1 + \alpha \mathbf{u}_2 + \beta \mathbf{u}_3 : \gamma + \alpha + \beta = 1\}
\]

Definition: A linear combination \(\gamma \mathbf{u}_1 + \alpha \mathbf{u}_2 + \beta \mathbf{u}_3\) where \(\gamma + \alpha + \beta = 1\) is an affine combination.
Affine combination

Definition: A linear combination

\[ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n \]

where

\[ \alpha_1 + \alpha_2 + \cdots + \alpha_n = 1 \]

is an affine combination.

Definition: The set of all affine combinations of vectors \( \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \) is called the affine hull of those vectors.

Affine hull of \( \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n = \mathbf{u}_1 + \text{Span} \{ \mathbf{u}_2 - \mathbf{u}_1, \ldots, \mathbf{u}_n - \mathbf{u}_1 \} \)

This shows that the affine hull of some vectors is an affine space.
Geometric objects not containing the origin: equations

Can express a plane as \( \mathbf{u}_1 + \mathcal{V} \) or affine hull of \( \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n \).

More familiar way to express a plane: as the solution set of an equation \( ax + by + cz = d \).

In vector terms,

\[
\{ [x, y, z] : [a, b, c] \cdot [x, y, z] = d \}
\]

In general, a geometric object (point, line, plane, ...) can be expressed as the solution set of a system of linear equations.

\[
\{ \mathbf{x} : \mathbf{a}_1 \cdot \mathbf{x} = \beta_1, \ldots, \mathbf{a}_m \cdot \mathbf{x} = \beta_m \}
\]

Conversely, is the solution set an affine space?

Consider solution set of a contradictory system of equations, e.g. \( 1x = 1, 2x = 1 \):

- Solution set is empty....
- ...but a vector space \( \mathcal{V} \) always contains the zero vector,
- ...so an affine space \( \mathbf{u}_1 + \mathcal{V} \) always contains at least one vector.

Turns out this the only exception:

**Theorem:** The solution set of a linear system is either empty or an affine space.
Affine spaces and linear systems

**Theorem:** The solution set of a linear system is either empty or an affine space.

Each linear system corresponds to a linear system with zero right-hand sides:

\[
\begin{align*}
\mathbf{a}_1 \cdot \mathbf{x} &= \beta_1 \\
\vdots \\
\mathbf{a}_m \cdot \mathbf{x} &= \beta_m
\end{align*}
\]

\[
\begin{align*}
\mathbf{a}_1 \cdot \mathbf{x} &= 0 \\
\vdots \\
\mathbf{a}_m \cdot \mathbf{x} &= 0
\end{align*}
\]

**Definition:**
A linear equation \( \mathbf{a} \cdot \mathbf{x} = 0 \) with zero right-hand side is a *homogeneous* linear equation. A system of homogeneous linear equations is called a *homogeneous* linear system.

**We already know:** The solution set of a homogeneous linear system is a vector space.

**Lemma:** Let \( \mathbf{u}_1 \) be a solution to a linear system. For any other vector \( \mathbf{u}_2 \), \( \mathbf{u}_2 \) is also a solution if and only if \( \mathbf{u}_2 - \mathbf{u}_1 \) is a solution to the corresponding homogeneous linear system.
Affine spaces and linear systems

\[
\begin{align*}
  a_1 \cdot x &= \beta_1 \\
  \vdots \\
  a_m \cdot x &= \beta_m \\
  a_1 \cdot x &= 0 \\
  \vdots \\
  a_m \cdot x &= 0
\end{align*}
\]

**Lemma:** Let \( u_1 \) be a solution to a linear system. For any other vector \( u_2 \), \( u_2 \) is also a solution if and only if \( u_2 - u_1 \) is a solution to the corresponding homogeneous linear system.

**Proof:** We assume \( a_1 \cdot u_1 = \beta_1, \ldots, a_m \cdot u_1 = \beta_m \), so

\[
\begin{align*}
  a_1 \cdot u_2 &= \beta_1 & a_1 \cdot u_2 - a_1 \cdot u_1 &= 0 & a_1 \cdot (u_2 - u_1) &= 0 \\
  \vdots & \quad \text{iff} \quad \vdots & \quad \text{iff} \quad \vdots \\
  a_m \cdot u_2 &= \beta_m & a_m \cdot u_2 - a_m \cdot u_1 &= 0 & a_m \cdot (u_2 - u_1) &= 0
\end{align*}
\]

QED
**Lemma:** Let $u_1$ be a solution to a linear system. For any other vector $u_2$, $u_2$ is also a solution if and only if $u_2 - u_1$ is a solution to the corresponding homogeneous linear system.

We use this lemma to prove the theorem:

**Theorem:** The solution set of a linear system is either empty or an affine space.

- Let $\mathcal{V} =$ set of solutions to corresponding homogeneous linear system.
- If the linear system has no solution, its solution set is empty.
- If it does has a solution $u_1$ then

$$\{\text{solutions to linear system}\} = \{u_2 : u_2 - u_1 \in \mathcal{V}\}$$

(substitute $v = u_2 - u_1$)

$$= \{u_1 + v : v \in \mathcal{V}\}$$

QED