The Pumping Lemma

How can you prove a language is not regular?

"L is regular" $\iff$ 
1. DFA that accepts $L$
2. NFA that accepts $L$
3. Regular expression that accepts $L$

$B = \{ a^n b^n \mid n \geq 0 \}$

$C = \{ w \mid w$ has an equal # of 0's and 1's $\}$

$D = \{ w \mid w$ has an equal # of occurrences of $01$ and $10$ $\}$

Pumping Lemma: If $A$ is a regular language, there exists $p$ s.t. for strings $s \in A$, $|s| \geq p$, $s$ may be divided into $s = xyz$

1. $x \subseteq \Sigma^*$, $y \neq \epsilon$
2. $|xy| \leq p$
3. $|y| > 0$

1. $L = a^*$, $p = 1$

$s = a^3$, $x = \epsilon$, $y = a$, $z = a^2$

2. $\epsilon a, ab, abab^3$, $p = 5$

As $s \in L$ s.t. $|s| \leq 5$, conditions 0, 0, 0 apply
3. P.L. is not $\iff$ (iff)

True: regular $\implies$ $\exists p$ s.t. $A_1 \ldots A_j$ can be pumped

False: $\exists p$ s.t. $A_1 \ldots A_j$ can be pumped $\not\implies$ regular

Proof: Let $A$ be regular

Let $M = (Q, \Sigma, S, q_0, \delta, F)$ be a DFA that accepts $A$

Let $p = |Q|$

Let $s \in A$ be any string s.t. $|s| \geq p$

Let $s_1, s_2, \ldots, s_n$

List the states that $M$ visits on input $s$

$s(\delta(s_i, a)) \Rightarrow s_{i+1}$

$A = s_1 \ldots s_n$

The sequence has length $n+1$, and $n+1 \geq p+1$

by pigeonhole principle that must be a repeating state $s_0 \ldots s_i s_0 \ldots s_i \ldots$
consider the first state that repeats
label the first appearance $s_{first}$
second appearance $s_{second}$

let

$x = s_1 \ldots s_{first-1}$
$y = s_{first} \ldots s_{second-1}$
$z = s_{second} \ldots s_n$

check $xyz$ satisfy the conditions 1, 2, 3

1. $\sqrt{y}$ because we found a cycle in the DFA, we can take it 0 or more times
2. $\sqrt{y}$ includes $s_{first}$, so it has nonzero length
3. $\sqrt{b/c}$ we chose first set of repeating states

$B = \{0^n1^n \mid n \geq 3 \}$ not regular

proof by contradiction
Assume $B$ is reg. then there exists
3 some $p$ s.t. $A \in B$, $1s \geq p$ s can be pumped
Consider $0^p \cdot 1^p$.

Suppose (1) $y$ has only 0's then $xy^2z$ has more 0's than 1's.

(2) $y$ has only 1's then $xy^2z$ has more 1's than 0's.

(3) $y$ has 0's and 1's.

\[ 0 \ldots 0 \ldots 0 \ldots 1 \ldots 1 \]

\[ y \]

Then $xy^2z$ is not in correct form to be in $B$.

Then $y$ can't exist, so $B$ not regular.

$c = \{ w \mid w \text{ has an equal number of 0's and 1's} \}$

Try: $s = (01)^p$ then $x = z$.

Then $y = 01$.

$z = (01)^{p-1}$.

This doesn't work, we have to find another example.
Try \( s = 0^n 1^n \)

In order to pump, we must have \( s = xy^2z \)

where \( y \) has an equal # of 0's and 1's

\[
D = \left\lfloor \frac{n}{3} \right\rfloor 
\]

for \( n \geq 0 \)

\[
= \prod_{i=1}^{n-1} i 
\]

for \( n > 0 \)

Not regular: suppose it is, \( w \) with pumping length \( p \)

Try: \( s = 1 \)

if we can write \( s \) as

\[
s = xy^2z, \quad \quad \text{underlined } x \neq y^2 \]

then \( xy^2z \in D \)

but \( |xy^2z| - |xyz| = |y| \leq |xy| < p \)

Thus

\[
\prod_{i=1}^{p+1} i = \prod_{i=3}^{p+1} i
\]

But the next smallest string in \( D \) after \( 1 \prod_{i=3}^{p+1} i \) which is longer than

\( xy^2z \), contradiction