\[ A_{TM} = \{ <M, w> \mid M \text{ is a TM that accepts } w \} \]

Last time:

Thm: \( A_{TM} \) is undecidable

\[ \text{Anna}_{TM} = \{ <M> \mid L(M) = \text{ empty set} \} \]

wts \( \text{Anna}_{TM} \) is undecidable

Pf: suppose Anna had a decider \( D_{\text{anna}} \)

Consider the following decider \( D_A \) for \( A_{TM} \)

\[ D_A: \text{ on input } <M, w> \]

\[ \text{compute } <M'> \text{ s.t.} \]

\[ <M, w> \in A_{TM} \iff <M'> \in \text{Anna}_{TM} \]

reduce \[ A_{TM} \] to \[ \text{Anna}_{TM} \]

Run \( D_{\text{anna}} \) on input \( <M'> \)

accept if it accepts

reject otherwise

Description of \( M' \):

on input \( x \), run \( M \) on input \( w \)

if \( M \) accepts and \( x = \text{"anna"} \), accept

else reject
Analysis of $DA_{\neg wts}$ that $DA$ is a decider for $A_{TM}$

Suppose $<M, w> \in A_{TM}$

Then $L(M') = \{\text{accepts}\}$, then $<M'> \in \text{Ann}.

And so $\text{Danna}$ accepts $<M'>$. So $DA$ accepts.

Suppose $<M, w> \notin A_{TM}$

Then $M'$ never accepts any $x$, so $L(M') = \emptyset$.

So $<M'> \notin \text{Ann}_{TM}$, so $\text{Danna}$ rejects $<M'>$. So $DA$ rejects.

Rice's Theorem

Let $P$ be any language that consists of description of TMs s.t.

1. $\forall M_1, M_2$ s.t. $L(M_1) = L(M_2)$
   $<M_1> \in P \Leftrightarrow <M_2> \in P$

2. $P$ is non-trivial, i.e. $\exists M$ s.t.
   $<M> \in P$ and $\exists M' s.t. <M'> \notin P$

Then $P$ is undecidable.
Let $mp$ be a TM such that $<mp> \in P$.

Let $M_{rej}$ be a TM that always rejects.

Suppose that $<M_{rej}> \notin P$.

Proof: $P$ is undecidable.

Suppose $P$ has a decider $D_p$.

Consider the following decider $D_A$ for $A_{TM}$.

$D_A$: An input $<M, w>$

1. Compute the description of TM $M'$
2. $M' = \text{on input } x$
3. $P_{M'} = \text{run } M \text{ on input } w$
4. If $M$ accepts, run $M_p$ on input $x$. If it accepts, accept. Else reject.

Run $D_p$ on input $<M'>$.
accept if it accepts
reject otherwise.

Analysis: $wts$ $D_A$ decides $A_{TM}$.

If $<M, w> \in A_{TM}$ then $L(M') = L(M_p)$ so $<M'> \in P$ so $D_p$ accepts so $D_A$ accepts.

If $<M, w> \notin A_{TM}$ then $L(M') = L(M_{rej})$ so $<M'> \notin P$ so $D_p$ rejects so $D_A$ rejects.

Therefore $D_A$ is a decider for $A_{TM}$, contradiction.
Lemma: $L$ undecidable $\iff L'$ undecidable

Proof: straightforward

Now suppose $\langle M_{rej}\rangle \notin P$, then we just proved that $P^c$ undecidable. Then $P$ is undecidable.

Undecidable languages:

- $\text{CFG}_{TM} = \{ <M> | L(M) \text{ is context free} \}$
- $\text{HALT}_{TM} = \{ <M, w> | M \text{ halts on input } w \}$
- $\text{Reg}_{TM} = \{ <M> | L(M) \text{ is regular} \}$
- $\text{E}_{TM} = \{ <M> | L(M) = \emptyset \}$
- $\text{EQ}_{TM} = \{ <M_1, M_2> | L(M_1) = L(M_2) \}$

All these languages have to do with TMs.

Undecidable languages unrelated to TMs:

- $\text{ALL}_{CFG} = \{ G \mid L(G) = \Sigma^* \}$
- $\text{PCP} = \{ \langle P \rangle \mid P \text{ is a collection of tiles} \}

s.t. \exists a sequence (seps allowed) of tiles s.t.

- Top string corresponds to bottom string