HW9
Due: Nov 15

Attach a fully filled-in cover sheet to the front of your printed homework. Your name should not appear anywhere except on the cover sheet; each individual page of the homework should include your banner ID instead.

While collaboration is encouraged in this class, please remember not to take away notes from any collaboration sessions. Also please list the names and logins of any of your collaborators at the beginning of each homework.

Please monitor Piazza, as we will post clarifications of questions there. You should hand in your solutions by 12:55 to the CSCI 51 bin on the second floor of the CIT. Late homeworks are not accepted.

Problem 1

After all the trouble with tribbles,\(^1\) Dr. McCoy has been tasked with removing all the deceased tribbles from the sickbay. The set \(S = \{x_1, x_2, \ldots, x_n\}\), contains the sizes of the tribbles he is throwing out, and \(x_i\) is the volume of the \(i\)th tribble. First, McCoy loads his items into bags, which he then brings to the transporter room, where Scotty beams them onto a nearby Klingon vessel. Each bag can only fit a volume of \(b\).\(^2\) Hence the sum of the volumes of the tribbles Dr. McCoy throws into each bag must be less than or equal to \(b\). McCoy wants to know if he can move all the tribbles from the sickbay to the transporter room in \(k\) or fewer bags, and since he is a “doctor, not a theoretical computer scientist,” he wants your help. Luckily, tribbles’ volumes only come in integer increments (space is a weird place).

Prove that McCoy’s problem is NP-Complete by a reduction from \textsc{SubsetSum}. You can assume without loss of generality that in the original \textsc{SubsetSum} problem, the target has a value that is at least half of the total sum of the values in the set (it may help to figure out why you can assume this).

\(^1\)TOS, Season 2, Episode 15
\(^2\)Tribbles are made mostly of liquid, and can easily be tightly packed in a bag.
Problem 2

After one too many warp-core, hyper-drive, and transporter malfunctions, Scotty decided it was time for the Enterprise to be sent to the shop. However, the ship came back with a surprising adjustment: it got a lot of new hallways equipped with single direction moving walkways. In fact, the Enterprise now has so many paths between any two points that it has become a problem, and Captain Kirk has not been able to report to the Sick Bay ever since the change, since he gets lost along the way. The ship’s architects have been tasked with reducing the number of hallways and rooms so that there are no more than one path between any two locations.

You are supposed to aid the architects in this, but you have a hunch that this problem is in fact NP-Complete. Prove this to get out of the job. Specifically,

let \( G = (V, E) \) be a directed graph with vertices \( V \) and edges \( E \). Define the language

\[
\text{HallwayConfusionControl} = \{ \langle G, k \rangle \mid \exists V' \subseteq V \text{ such that } |V'| \leq k, \text{ and } E' = \{(u, v) \in E \mid u \in V' \text{ or } v \in V'\}, \text{ and } G' = (V - V', E - E') \text{ is acyclic} \}
\]

Prove that HallwayConfusionControl is NP-Complete.

Problem 3

The crew has lost Sulu in the Enterprise and needs to find him as soon as possible for the next mission.

Fortunately, the team knows Sulu loves blueberry pie and the team is able to put up a limited number of slices of blueberry pie, and an equal number of cameras in different rooms in the ship. The team will find Sulu if he is lured into any room with a camera and a slice of blueberry pie.

The space ship \( G \) is a set of vertices and edges, where the vertices are rooms and the edges are between adjacent rooms. The team must choose a subset of the vertices \( A \) to put cameras and slices of blueberry pie in, but only have \( k \) cameras and \( k \) slices of blueberry pie and so can only put them in at most \( k \) rooms. The crew knows that Sulu will always take the shortest path from
wherever room he started in to the closest room with a slice of blueberry pie. They want to choose the set of vertices $A$, with $|A| = k$ so that if Sulu starts in a room with no camera, and then takes the shortest path to the closest room with a slice of blueberry pie, the average (over all possible starting rooms with no camera) number of rooms he must enter to reach a room with a camera is at most a fixed rational number $r$. The language $\text{FindSulu}$ consists of those triples $(G, k, r)$ for which this is possible.

Prove that $\text{FindSulu}$ is NP-Complete.

The following questions are lab problems.

**Lab Problem 1**

While traversing space, the Enterprise found a strange planet Choagie that is exclusively populated by sandwiches. These sandwiches are obsessed with family trees. A family tree is a directed graph, where each vertex represents a sandwich, and an edge $(u, v)$ means that $u$ is a biological parent of $v$ (equivalently, that $v$ is a child of $u$). Each vertex has either two parents, or none. The graph must be acyclic, because one cannot be one’s own ancestor. Finally, although the vertices of a family tree are not labelled with the biological sexes of the sandwiches they represent, sandwiches nevertheless have biological sexes, so there exists some assignment of biological sexes to the vertices such that every child has parents of distinct biological sexes.

In this problem, we look at the language $\text{Family-Tree}(g)$ that consists of digraphs representing valid family trees with $g$ biological sexes.

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3Recall that $r$ must be rational for it to be written down on a Turing machine’s tape.

4Space is really weird
a. For all $g$, prove that Family-Tree($g$) is in NP.

b. For $g \geq 3$, prove that Family-Tree($g$) is NP-complete by reducing from 3-colorability.

*Remember:* even though there are $g$ biological sexes, each child vertex has exactly two parents of different biological sexes, and the input doesn’t specify the biological sex of any vertices.
Lab Problem 2

While roaming the regions of space bordering the galactic barrier,\(^5\) the Enterprise picks up a signal that appears to have been transmitted a long time ago from a galaxy far, far away. Since the Enterprise itself cannot make it across the galactic barrier to visit neighboring galaxies, Spock is intrigued by the opportunity to hear from civilizations where no man has gone before. However, passing through the barrier has caused the signal to become distorted, so it is not easily decodable. Luckily, the gravitational pull of the barrier causes the original message to be repeated indefinitely with some slight error at every repetition. Spock plans to use this to decode the message by finding a string that is closest to the repetitions within some error bound.

Consider the language $\text{ClosestString}$ of a number $k$ and a set $S$ of $\{0, 1\}$ strings, all of length $n$ for some $n \geq 0$, such that there exists a string $w$, also of length $n$, which for each string $s \in S$ differs at no more than $k$ positions. You can think of $w$ as a signal that was sent, and the set $S$ as a set of received strings that have some error less than $k$. Specifically,

$$\text{ClosestString} = \{ (k, S) | \exists w : \forall s \in S : d(w, s) \leq k \}$$

where $d(w, s)$ denotes the number of positions at which $w$ and $s$ differ. For example, $\langle 2, \{0011, 1010, 0110, 1001\} \rangle \in \text{ClosestString}$, because all four strings differ from 1111 at no more than 2 positions.

a. Show that, for a given integer $x \geq 0$, there exists a set $B_x \subset \{0 \cup 1\}^{2x}$, such that for any string $w \in \{0 \cup 1\}^{2x}$:

$$w \in \{00 \cup 11\}^x \iff \forall b \in B_x : d(w, b) \leq x$$

**Hint.** for $x = 6$, consider the four strings:

- 01 01 01 01 01 01
- 10 01 01 01 01 01
- 10 10 10 10 10 10
- 01 10 10 10 10 10

\(^5\)The galactic barrier is a negative energy field surrounding our galaxy, the Milky Way. Check out Star Trek TOS S1E3, S2E22, S3E5 for some great plot twists and possibly contradictory information on this non-scientific phenomenon!
Try constructing $B_6$ using this set as a subset of $B_6$.

b. Show that $\text{CLOSEST}\text{STRING}$ is NP-hard by reduction from 3SAT.

c. Show that $\text{CLOSEST}\text{STRING}$ is NP-complete.

**Lab Problem 3**

The Enterprise’s drives recently malfunctioned, and took most of the communicator network with it. The crew want to split up into smaller groups so that they can search a nearby planet for dilithium crystals, but they want to stay in touch, and each person’s communicator can only reach a few other communicators.

We can express this problem as the language

$$\text{COMMUNICATOR}\text{CONNECTION} = \{ (G, k) \mid \exists \text{ a partition } \{V_1, \ldots, V_k\} \text{ of } V \text{ such that } \forall v \in V_i, \forall V_j \text{ where } i \neq j, \exists u \in V_j : (v, u) \in E \}$$

Kirk wants Spock to divide up the crew into as many teams as possible, while still ensuring that every member of a group can contact at least one member of every other group, but Spock insists that just figuring out whether the team can be split up into $k$ groups is NP-complete, and thus trying to find the optimal solution would be illogical. Help him prove this, so that he can show that good communication between the teams must be sacrificed for the collective good of the ship.

*(Hint: reduce to this from 3-colorability.)*