Last time

Constructive proofs, Existential Proof

Proposition: \( \exists x, y \in \mathbb{R} \setminus \mathbb{Q} \text{ st } xy \in \mathbb{Q} \)

\textit{Proof.} Recall \( \sqrt{2} \) is irrational
Consider: \( \sqrt{2} \sqrt{2} \)

\textbf{Case 1:} \( \sqrt{2} \sqrt{2} \) is rational, in which case we are done

\textbf{Case 2:} \( \sqrt{2} \sqrt{2} \) is irrational
Then let \( x = \sqrt{2} \sqrt{2}, y = \sqrt{2}, xy = 2 \)

\textbf{Cartesian Product}

Let \( A, B \) be sets. Define \( A \times B = \{(a, b) | a \in A \text{ and } b \in B \} \).
Note \((a, b)\) is an ordered pair.

\textbf{Ex:} \( A = \{1, 2, a\} \) and \( B = \{b, 2, 3\} \)
\( (1, b), (1, 2), (2, 2), (a, 3) \in A \times B \)
\( (b, 2) \notin A \times B \)

\textbf{Ex:} \( \mathbb{R} \times \mathbb{R} = \mathbb{R}^2 \) This is all the points on the plane
\( \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2 \) This is all integer coordinates on the plane
\( \mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R} = \mathbb{R}^n \) \( n \)-fold Cartesian product

\textbf{Binary Strings as Cartesian Products}

\( S = \{0, 1\} \)
\( S \times S \times S \times \ldots \times S = S^n = \text{Binary strings of length } n \)

\textbf{Relations}

A relation \( R \) on \( A \times B \) is any subset of \( A \times B, R \subseteq A \times B \)
A relation \( R \subseteq A \times A \) is called a “relation on \( A \)”
Ways to denote that a pair \((a, b)\) is in relation \( R \): \((a, b) \in R \) or \( aRb \)
Properties of Relations

Let $R$ be a relation on $A$

1. **Reflexive:** $R$ is called reflexive if $\forall a \in A, (a, a) \in R$

2. **Symmetric:** $R$ is called symmetric if $\forall a, b \in A, (a, b) \in R \implies (b, a) \in R$

3. **Transitive:** $R$ is called transitive $\forall a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R \implies (a, c) \in R$

**Equivalence Relation:** a relation that is reflexive, symmetric and transitive