Last time

Constructive proofs, Existential Proof

Proposition: \( \exists x, y \in \mathbb{R} \setminus \mathbb{Q} \text{ st } x^y \in \mathbb{Q} \)

Proof. Recall \( \sqrt{2} \) is irrational Consider: \( \sqrt{2}^{\sqrt{2}} \)

Case 1: \( \sqrt{2}^{\sqrt{2}} \) is rational, in which case we are done

Case 2: \( \sqrt{2}^{\sqrt{2}} \) is irrational

Then let \( x = \sqrt{2}^{\sqrt{2}}, y = \sqrt{2}, x^y = 2 \)

\[ \square \]

Cartesian Product

Let \( A, B \) be sets. Define \( A \times B = \{(a, b) | a \in A \text{ and } b \in B\} \).

Note \( (a, b) \) is an ordered pair.

Ex: \( A = \{1, 2, a\} \) and \( B = \{b, 2, 3\} \)

\( (1, b), (1, 2), (2, 2), (a, 3) \in A \times B \)

\( (b, 2) \notin A \times B \)

Ex: \( \mathbb{R} \times \mathbb{R} = \mathbb{R}^2 \) This is all the points on the plane

\( \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2 \) This is all integer coordinates on the plane

\( \mathbb{R} \times \mathbb{R} \ldots \times \mathbb{R} = \mathbb{R}^n \) n-fold Cartesian product

Binary Strings as Cartesian Products

\( S = \{0, 1\} \)

\( S \times S \times S \times \ldots \times S = S^n = \text{Binary strings of length } n \)

Relations

A relation \( R \) on \( A \times B \) is any subset of \( A \times B \), \( R \subseteq A \times B \)

A relation \( R \subseteq A \times A \) is called a “relation on \( A \)”

Ways to denote that a pair \( (a, b) \) is in relation \( R \): \( (a, b) \in R \) or \( aRb \)
Properties of Relations

Let $R$ be a relation on $A$

1. **Reflexive**: $R$ is called reflexive if $\forall a \in A, (a, a) \in R$

2. **Symmetric**: $R$ is called symmetric if $\forall a, b \in A, (a, b) \in R \implies (b, a) \in R$

3. **Transitive**: $R$ is called transitive $\forall a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R \implies (a, c) \in R$

Ex 1: $R$ on $\mathbb{Z}$ $R \subseteq \mathbb{Z}$

≤ Relation: $R = \{(a, b) | a \leq b\}$, Reflexive and Transitive, but not symmetric

Ex 2: $R$ on $\mathbb{Z}$ Parity Relation: $R = \{(a, b) | \text{a and b are both even or a and b are both odd}\}$

Reflexive, Symmetric and Transitive - This is an equivalence relation

**Equivalence Relation**: a relation that is reflexive, symmetric and transitive

Equivalence relations create **partitions**

**Equivalence Class**

**Defn**: $R$ on $A$, $[a]_R = \{x \in A | (x, a) \in R\}$. $[a]_R$ is the equivalence class of $a$.

**Example**:

$R$ parity on $\mathbb{Z}$

$[2]_{\text{Parity}} = \{\ldots -4, -2, 0, 2, 4, \ldots\}$

$[2]_{\text{Parity}} = [8]_{\text{Parity}} = \text{[Any even number]}$\text{Parity} $[3]_{\text{Parity}} = \{\ldots -3, -1, 1, 3, \ldots\}$

We have now examined all equivalence classes in this relation. There are 2 equivalence classes for Parity. These equivalence classes **partition** all of the integers. Every integer is either even or odd but not both. It breaks up the integers into two disjoint subsets.

**Partitions**

**Defn**: A partition of set $A$ is a collection of disjoint subsets of $A$ that cover $A$. So, $P = B_1B_2\ldots B_n$ where $B_i \subseteq A$

- $\forall a \in A, a \in B$ for some $i$
• \( \forall i, j B_i \cap B_j = \emptyset \)

**Prop:** Given an eq relation on \( R \) on a set \( A \), the eq classes partition \( A \).

Proof. Let \( R \) be eq relation on \( A \). Consider all **distinct** equiv classes. (distinct = not equal as a set).

1. \( \forall a \in A, (a, a) \in R \implies a \in [a]_R \) (bc Reflexive)

2. Let \( B_1, B_2 \) be distinct so \( B_1 \neq B_2 \)

   Claim: \( B_1 \cap B_2 = \emptyset \)
   Assume for the sake of contradiction that \( B_1 \cap B_2 \neq \emptyset \)

   Let \( x \in B_1 \cap B_2 \)
   \( B_1 = [y]_R \) and \( B_2 = [z]_R \)

   Let \( w \in B_1 \implies (w, y) \in R \) \( x \in B_1 \implies (x, y) \in R \implies (y, x) \in R \implies (w, x) \in R \)

   \( x \in B_2 \implies (x, z) \in R \implies (w, z) \in R \implies w \in B_2 \)

   So \( B_1 \subseteq B_2 \)

   Note that that proof to show \( B_2 \subseteq B_1 \) is exactly the same with \( B_1 \) and \( B_2 \) reversed

\[ \square \]

**Example:** Fractions in lowest terms

\( R \) on \( Q \) so \( R \subseteq Q \times Q \)

\( (\frac{a}{b}, \frac{x}{y}) \in R \) iff \( a \cdot y = b \cdot x \)

\([\frac{1}{2}]_R \) contains: \( \frac{2}{4}, \frac{4}{8}, \frac{50}{100} \)