Induction

\( P(n) \forall n \geq b \), \( n \in \mathbb{Z} \)

- Base Case: \( p(b) \)
- Inductive step: \( p(k) \implies p(k+1) \)

**Ex 1** \( p(n) : 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \)

*Proof.* Base Case: \( p(1) \)
\[
1 = \frac{1(1+1)}{2}
\]

Inductive step: Assume \( p(k) \) is true for a fixed but arbitrary value \( k \)
Claim: Want to show \( p(k+1) \) is true

*Proof.*
\[
p(k+1) : 1 + 2 + 3 + \ldots + k + k + 1 = \frac{k(k+1)}{2} + \frac{2 \cdot (k+1)}{2} \quad \text{by Inductive Hypothesis}
\]
\[
= \frac{(k+1)(k+2)}{2}
\]

**Ex 2:** \( p(n) \): number of subsets of an \( n \) element set = \( 2^n \) for \( n > 0 \)
\( P(0) : 1 = 2^0 = \text{number of subsets of } \emptyset \)
\( P(1) : 2 = 2^1 = \text{number of subsets of a 1 element set} \)

Inductive step: Assume \( P(k) \) for a fixed but arbitrary value of \( k \)
Claim: \( P(k+1) \) is true. Let \( X \) be a set with \( k+1 \) elements. Let \( y \in X \)

How many subsets of \( X \) contain \( y \)?
y plus any subset of \( X \setminus \{y\} \)
By our I.H., there are \( 2^k \) subsets of \( X \setminus \{y\} \) so there are still \( 2^n \) when we add \( y \)

How many subsets of \( X \) do no contain \( y \)?
Any subset of \( X \setminus \{y\} \)
By our I.H., there are \( 2^k \) subsets of \( X \setminus y \)
Total number of subsets = \( 2^k + 2^k = 2^{k+1} \)