Asymptotic Analysis

**Definition:** A function $f$ is $O(g)$ if $\exists c, x_0 \geq 0$ such that
\[ \forall x \geq x_0, \quad |f(x)| \leq c \cdot |g(x)| \]

This captures the idea that beyond a certain value, $x_0$, the function $g$ will provide an upper bound for $f$.

$O(g)$ is the set of all functions $f$ such that $f$ is $O(g)$. Formally, $f \in O(g)$; however, this idea is often denoted “$f = O(g)$”.

**Example:** $f(x) = x^2 + 4x + 15$

If $x > 1$ then $x < x^2$. \[ \Rightarrow \]
If $x > 1$ then $x^2 + 4x + 15 < x^2 + 4x^2 + 15x^2 = 20x^2$ \[ \Rightarrow \]

$f$ is $O(x^2)$

Setting $x_0 = 1$ and $c = 20$, we see that $f$ is $O(x^2)$. Note that these constants are not unique. We can also go the other direction and say that $x^2$ is $O(f(x))$.

**Definition:** A function $f$ is $O(g)$ iff $g$ is $\Omega(f)$. This provides $f$ as a lower bound for $g$.

**Definition:** A function $f$ is $\Theta(g)$ iff $f$ is $O(g)$ and $f$ is $\Omega(g)$.

Note: this is often what people mean when they say that $f$ is $O(g)$. Be cognizant of the distinction between these two terms!

**Example:** $f(x) = x^2 + 4x + 15$

There are infinitely many upper bounds on $f$; for example, we have already shown that $f$ is $O(x^2)$, but $f$ is also $O(x^3)$, and $O(x^{100})$, etc.

However, we can differentiate between these bounds by looking at $\Theta$. $f$ is $\Theta(x^2)$, but $f$ is not $\Theta(x^3)$.

**Example:** $x^2$ is not $O(x)$.

Let’s prove this by contradiction. Suppose $x^2 \in O(x)$. Then $\exists x_0, c$ such that $x^2 \leq c \cdot x \Rightarrow x \leq c \forall x > x_0$. This is a contradiction because $c$ is a constant, so $x^2 \not\in O(x)$.

**Example:** $f(n) = 1 + 2 + \ldots + n = \frac{n(n+1)}{2} = \binom{n+1}{2}$

**Claim:** $f(n)$ is $O(n^2)$.

**Proof.** $1 + 2 + 3 + \ldots + n \leq n + n + n + \ldots + n = n^2$. Setting $c = 1, x_0 = 1$ we have that $f$ is $O(n)$.

**Claim:** $f(n)$ is $\Omega(n^2)$. 


Proof. $1 + 2 + 3 + \ldots + n \geq \left\lceil \frac{n}{2} \right\rceil + \left( \left\lceil \frac{n}{2} \right\rceil + 1 \right) + \left( \left\lceil \frac{n}{2} \right\rceil + 2 \right) + \ldots + n > \left\lceil \frac{n}{2} \right\rceil \cdot \left\lceil \frac{n}{2} \right\rceil \geq \frac{n}{2} \cdot \frac{n}{2} = \frac{n^2}{4}$. Thus, $n^2$ acts as a lower bound for $f$ and we have that $f$ is $\Omega(n)$. 

Claim: $f(n)$ is $\Theta(n^2)$.

Proof. As shown above, $f$ is $O(n^2)$ and $f$ is $\Omega(n^2)$. Therefore $f$ is $\Theta(n^2)$. 

Notice that in general, for a binomial coefficient of the form \( \binom{n}{k} \), its growth rate will be $n^k$. 