Probability Cont.

Ex. Suppose we have a random binary string of length $n$: $\{0,1\}^n$. Let $p$ represent the probability that the $i^{th}$ bit is a 1. We can define the following random variables:

- $X(\omega) =$ total number of 1’s in $\omega$
- $Y(\omega) = 2 \cdot (\text{number of 1’s in } \omega) - 1 \cdot (\text{number of 0’s in } \omega)$
- $Z(\omega) = 7$

Note that there are infinitely many possible random variables to define.

Ex. Suppose $n = 1$. Using the random variables defined in the previous example, what is the expectation of the random variable $X$, $E[X]$?

Using the definition of expectation, we see

\[
E[X] = \sum_{\omega \in S} p(\omega) \cdot X(\omega)
= p(0) \cdot X(0) + p(1) \cdot X(1)
= (1 - p) \cdot 0 + p \cdot 1
= p
\]

What if $n = 2$? In this case, our sample space $S$ is $S = \{00, 01, 10, 11\}$ and the expectation can be determined as follows:

\[
E[X] = \sum_{\omega \in S} p(\omega) \cdot X(\omega)
= p(00) \cdot X(00) + p(10) \cdot X(10) + p(01) \cdot X(01) + p(11) \cdot X(11)
= (1 - p)^2 \cdot 0 + p \cdot (1 - p) \cdot 1 + (1 - p) \cdot p \cdot 1 + p^2 \cdot 2
\]

For example, if $p = 1/2$, our expectation is $\frac{1}{4} + \frac{1}{4} + \frac{2}{4} = 1$.

Linearity of Expectation

Suppose we have a random variable $X$ that can be broken into the sum of other random variables:

\[X = X_1 + X_2 + \ldots + X_n\]
Then the expectation of $X$ can be written as follows:

$$E[X] = \sum_{\omega \in S} p(\omega)X(\omega)$$

$$= \sum_{\omega \in S} p(\omega)(X_1(\omega) + ... + X_n(\omega))$$

$$= \sum_{\omega \in S} \left(p(\omega)X_1(\omega) + ... + p(\omega)X_n(\omega)\right)$$

$$= \sum_{\omega \in S} p(\omega)X_1(\omega) + ... + \sum_{\omega \in S} p(\omega)X_n(\omega)$$

$$= E[X_1] + ... + E[X_n]$$

**Indicator Random Variables**

Suppose again that we have a binary string of length $n$, $\{0, 1\}^n$. Let $X(\omega)$ be the number of 1’s in $\omega$. We can then define

$$X_i = \begin{cases} 
1 & \text{if the } i^{th} \text{ bit is a 1} \\
0 & \text{otherwise}
\end{cases}$$

Then we have

$$X = \sum_{i=1}^{n} X_i$$

and using linearity of expectation, we can determine the expectation of $X$ as follows:

$$E[X] = E[X_1 + X_2 + ... + X_n]$$

$$= E[X_1] + E[X_2] + ... + E[X_n]$$

$$= p + p + ... + p$$

$$= n \cdot p$$

**Ex. Random Graphs**

Suppose we have a random graph with vertices $V = \{1, 2, ..., n\}$, and that we include each edge with probability $p$.

**Question:** What is the expected number of triangles in this model?

We define a random variable $T : S \rightarrow \mathbb{R}$ such that $T(\omega) =$ number of triangles in $\omega$. We can use the following random variable:

$$T_i = \begin{cases} 
1 & \text{if the } i^{th} \text{ triangle is in } \omega \\
0 & \text{otherwise}
\end{cases}$$
We have \( \binom{n}{3} \) possible triangles, so our random variable \( T \) can be written \( T = T_1 + ... + T_{\binom{n}{3}} \). Then

\[
E[T] = \sum_{i=1}^{\binom{n}{3}} E[T_i]
\]

\[
= \sum_{i=1}^{\binom{n}{3}} p^3 \cdot 1 + 0
\]

\[
= \binom{n}{3} \cdot p^3
\]

**Ex. 3-SAT Random Assignment** Consider a proposition with \( m \) clauses of the following form:

\[
(x_1 \lor \neg x_2 \lor x_7) \land \\
(x_3 \lor \neg x_6 \lor x_5) \land \\
(\neg x_7 \lor x_3 \lor \neg x_1) \land ...
\]

Let’s assign a 0 or a 1 to each \( x_i \) at random with probability \( p \). Let \( C \) be the random variable that represents the number of clauses that are satisfied. \( C \) can then be broken up using the following indicator random variable:

\[
C_i = \begin{cases} 
1 & \text{if the } i^{th} \text{ clause is satisfied} \\
0 & \text{otherwise}
\end{cases}
\]

Note that

\[
p(C_i = 1) = 1 - p(C_i = 0) = 1 - p^3
\]

So we have

\[
E[C] = (1 - p^3) \cdot m
\]

If we assign 0/1 with probability \( p = 1/2 \) then we expect to satisfy 7/8 of our clauses.