Graph Theory

A graph can be thought of as a “model of pairwise relations”.

Def: A graph $G = (V, E)$ is a pair of objects $V$ and $E$ such that $V$ is a finite set and $E$ is a collection of pairs of elements in $V$.
$V$ is known as the set of “vertices”, and $E$ is the set of “edges”.

Representation

Consider the following graph $G$ such that

\[
G = (V, E) \\
V = \{1, 2, 3, 4, 5, 6\} \\
E = \{\{1, 2\}, \{6, 1\}, \{4, 6\}, \{3, 4\}, \{3, 5\}, \{2, 5\}\}
\]

We can construct a visual representation of this graph by making points for each element in $V$ and adding lines between these points for each element in $E$:

For now, we are not allowing for cycles, directed pairs, or weighted edges.
Counting Graphs

**Question:** How many graphs are there on $n$ vertices?

There are $\binom{n}{2}$ pairs of distinct points. Each of these pairs determines one possible edge, and a graph can be constructed from any subset of those possible edges. Therefore, there are $2^{\binom{n}{2}}$ possible graphs.

Call two graphs *similar* if they differ only by a relabeling of the vertices.

**Question:** How many graphs are there up to similarity?

We cannot arrive at an exact solution, but we can approximate this number by dividing the total number of graphs by the number of permutations of the $n$ vertices:

$$\sim \frac{2^{\binom{n}{2}}}{n!}$$

Trees

**Def:** A graph $G$ is a tree if it is *connected* and *acyclic*.

- A graph is *connected* if for every pair of vertices $v, w \in V$ there exists a sequence of edges $e_1, ..., e_k$ such that $v \in e_1$, $w \in e_k$ and $e_i \cap e_{i+1} \neq \emptyset$ for all $i$.
- A graph is *acyclic* if there does not exist a sequence of distinct edges $e_1, ..., e_k$ such that $e_i \cap e_{i+1} \neq \emptyset$ for all $i$ and $e_1 \cap e_k \neq \emptyset$.

**Equivalent definitions of a tree:**

- $\forall v, w \in V$ there exists a unique path from $v$ to $x$.
- Connected and removing any edge $e \in E$ disconnects the graph (minimally connected).
- Acyclic and adding any edge forms a cycle (maximally acyclic).

With this in mind, we see that proving any two of the three properties of trees (connected, acyclic, $n - 1$ edges) is sufficient to show the third.

**Question:** How many trees are there on $n$ vertices?

This is equivalent to asking how many strings of length $n - 2$ we can form from an $n$-element set. Thus, the total number of trees is

$$n^{n-2}$$

This implies that there exists a bijection between trees on $\{1, ..., n\}$ and strings of length $n - 2$ of the numbers $\{1, ..., n\}$. This is known as the **Prufer Code** of a tree.