Probability and Independence

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Overview

Independence (17.6)
  Alternative Formulation (17.6.1)
  Mutual Independence (17.6.3)
Conditional probability

**Definition:** The conditional probability of event $A$ given event $B$ is:

$$
Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}.
$$

Conceptually, if we limit ourselves to the outcomes in $B$, how likely is an outcome in $A$?

Example:

- **A:** Die shows a number divisible by 3. $Pr[A] = 1/3$. (3 and 6 from the six possibilities.)
- **B:** Die shows an odd number. $Pr[B] = 1/2$.
- What does $Pr[A \cap B]$ mean? Die shows an odd number divisible by 3. $Pr[A \cap B] = 1/6$ (only 3).
- What does $Pr[A|B]$ mean? Die shows a number divisible by 3 given that it’s odd. $Pr[A|B] = 1/3$ (probability of picking 3 from 1, 3, 5). Also, $\frac{1/6}{1/2} = \frac{1}{6} \times 2 = \frac{1}{3}$.
Independence

**Definition:** Event $A$ is *independent of* event $B$ iff

$$Pr[A|B] = Pr[A].$$

If $Pr[B] = 0$, we say it is independent of any other event including itself.

Example:

- **A:** Die shows the maximum or minimum number. $Pr[A] = 1/3$. (1, 6 from the six possibilities.)
- **B:** Die shows an odd number. $Pr[B] = 1/2$.
- $Pr[A|B] = 1/3$. (1 from 1,3,5.) So, $A$ and $B$ are independent.
- **C:** Die shows an even number. $Pr[C] = 1/2$.
Independent events multiply

**Theorem:** $A$ is independent of $B$ iff

$$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B].$$

**Proof:** By cases.

- **Case 1:** If $\Pr[A] = 0$ or $\Pr[B] = 0$, then $\Pr[A \cap B] = 0$. Equality and independence are both achieved.

- **Case 2:** Otherwise, $A$ is independent of $B$ iff $\Pr[A|B] = \Pr[A]$ by definition. Substituting in the definition of conditional probability, we have $\Pr[A|B] = \Pr[A]$ iff

  $$\frac{\Pr[A \cap B]}{\Pr[B]} = \Pr[A].$$

  Multiplying both sides by $\Pr[B]$, we have $\Pr[A \cap B]/\Pr[B] = \Pr[A]$ iff

  $$\Pr[A \cap B] = \Pr[A] \cdot \Pr[B].$$

  QED.
Mutual Independence

**Definition:** A set of events $E_1, E_2, \ldots, E_n$ is *mutually independent* iff for all subsets $S \subseteq [1, n],$

$$\Pr \left[ \bigcap_{j \in S} E_j \right] = \prod_{j \in S} \Pr[E_j].$$

Example: If we toss $n$ fair coins, the tosses are mutually independent iff for every subset of $m$ coins, the probability that every coin in the subset comes up heads is $2^{-m}$. 


### Independent Componuts

<table>
<thead>
<tr>
<th>Componuts</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Glaze jelly angel</td>
<td>0.028</td>
</tr>
<tr>
<td>Glaze jelly devil</td>
<td>0.042</td>
</tr>
<tr>
<td>Glaze creme angel</td>
<td>0.252</td>
</tr>
<tr>
<td>Glaze creme devil</td>
<td>0.378</td>
</tr>
<tr>
<td>Plain jelly angel</td>
<td>0.012</td>
</tr>
<tr>
<td>Plain jelly devil</td>
<td>0.018</td>
</tr>
<tr>
<td>Plain creme angel</td>
<td>0.108</td>
</tr>
<tr>
<td>Plain creme devil</td>
<td>0.162</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\text{Pr[glaze]} &= 0.028 + 0.042 + 0.252 + 0.378 = 0.7 \\
\text{Pr[jelly]} &= 0.028 + 0.042 + 0.012 + 0.018 = 0.1 \\
\text{Pr[angel]} &= 0.028 + 0.252 + 0.012 + 0.108 = 0.4 \\
\text{Pr[glaze and jelly]} &= 0.028 + 0.042 = 0.07 = 0.7 \times 0.1 \\
\text{Pr[glaze and angel]} &= 0.028 + 0.252 = 0.28 = 0.7 \times 0.4 \\
\text{Pr[jelly and angel]} &= 0.028 + 0.012 = 0.04 = 0.1 \times 0.4 \\
\text{Pr[glaze and jelly and angel]} &= 0.028 = 0.7 \times 0.1 \times 0.4
\end{align*}
\]
Pairwise independence isn’t mutual independence

It is natural to think pairwise independence implies mutual independence. If $a$ is independent of $b$ and $c$, and $b$ and $c$ are independent of each other, how could $a$, $b$, and $c$ not be independent??

Example: Tyler, Julia, and Julie each pick a bit. If I tell you the sum of Tyler and Julia’s bits (mod 2), do you learn anything about either of their bits? No.

But, what if I tell you two of these pairwise sums mod 2?

- Tyler + Julie 1
- Tyler + Julia 0
- Julia + Julie 1

Must be 1 because we know Tyler doesn’t match Julie, but Tyler and Julia match. So, Julia and Julie can’t match.

$k$-wise does not imply $(k + 1)$-wise mutual independence.