Binomial Theorem, Inclusion/Exclusion

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Overview

The Binomial Theorem (15.7)

Inclusion-Exclusion (15.12)

Union of Two Sets (15.12.1)
Union of Three Sets (15.12.2)
Sequences with 42, 04, or 60 (15.12.3)
Union of n Sets (15.12.4)
Historical perspective

*I’m very well acquainted too with matters mathematical,*
Historical perspective

I’m very well acquainted too with matters mathematical,
I understand equations, both the simple and quadratical,
Historical perspective

I’m very well acquainted too with matters mathematical,
I understand equations, both the simple and quadratical,
About binomial theorem I am teeming with a lot o’ news—
Historical perspective

*I’m very well acquainted too with matters mathematical, I understand equations, both the simple and quadratical, About binomial theorem I am teeming with a lot o’ news— With many cheerful facts about the square of the hypotenuse.*

From “Modern Major General” by Arthur Sullivan and William Schwenck Gilbert (1879)
Binomials to powers: Examples

\[(a + b)^2 = aa + ab + ba + bb\]
Binomials to powers: Examples

\[(a + b)^2 = \quad aa + ab + ba + bb\]
\[\quad = \quad a^2 + 2ab + b^2\]
Binomials to powers: Examples

\[(a + b)^2 = aa + ab + ba + bb\]
\[= a^2 + 2ab + b^2\]

\[(a + b)^3 = aaa + aab + aba + abb\]
\[+ baa + bab + bba + bbb\]
Binomial Theorem, Inclusion/Exclusion

The Binomial Theorem (15.7)

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\[(a + b)^2 = aa + ab + ba + bb\]
\[= a^2 + 2ab + b^2\]

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\[= a^3 + 3a^2b + 3ab^2 + b^3\]
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\[(a + b)^4 = \quad aaaa + aaab + aaba + aabb\]
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\[+ \quad baaa + baab + baba + babb\]
\[+ \quad bbaa + bbab + bbba + bbbb\]
Binomial Theorem, Inclusion/Exclusion

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\((a + b)^2 = aa + ab + ba + bb\)
\[= a^2 + 2ab + b^2\]

\((a + b)^3 = aaa + aab + aba + abb + baa + bab + bba + bbb\)
\[= a^3 + 3a^2b + 3ab^2 + b^3\]

\((a + b)^4 = aaaa + aaab + aaba + aabb + abaa + abab + abba + abbb + baaa + baab + baba + babb + bbaa + bbab + bbaa + bbbb\)
\[= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\]
Binomials to powers: Examples

\[(a + b)^2 = aa + ab + ba + bb = a^2 + 2ab + b^2\]

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How about \((a + b)^n\)? How many terms consist of exactly \(k\) bs?
Binomials to powers: Examples

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\[= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4\]

How about \((a + b)^n\)? How many terms consist of exactly \(k\) \(b\)s?
Since it’s all combinations of an \(a\) and \(b\) in each position, there are \(\binom{n}{k}\) such terms.
Binomial theorem

**Theorem**: For all $n \in \mathbb{N}$, $a, b \in \mathbb{R}$,

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$
Binomial theorem

**Theorem:** For all \( n \in \mathbb{N}, a, b \in \mathbb{R} \),

\[
(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.
\]

Sometimes \( \binom{n}{k} \) called the *binomial coefficient* because of this connection.
Pets and sets

\(S\): Set of all students in CS22.
Pets and sets

\[ S: \text{Set of all students in CS22.} \]
\[ D \subseteq S: \text{Set of all students in CS22 who have a pet dog.} \]
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$D \cup C$:
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$|D \cup C| = |D| + |C|?$
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$|D \cup C| = |D| + |C|?$ Handles people who have neither correctly.
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$|D \cup C| = |D| + |C|$? Handles people who have neither correctly. Handles people who have one pet correctly.
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$|D \cup C| = |D| + |C|?$ Handles people who have neither correctly. Handles people who have one pet correctly. Messes up on people who have both.
Formulas for union

What's wrong with each formula for $|C \cup D|$?
Formulas for union

What's wrong with each formula for $|C \cup D|$?

▶ $|C| + |D|$?

Double counted people who have both.

Skipped people who have both.

Actually, that should work.

But, set difference can be tricky.

Nailed it. Correct for double counting.
Formulas for union

What’s wrong with each formula for $|C \cup D|$?

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Formulas for union

What's wrong with each formula for $|C \cup D|$?

- $|C| + |D|$? Double counted people who have both.
- $|C - D| + |D - C|$?
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strupai

$\Rightarrow |C| + |D|$? Double counted people who have both.

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$\Rightarrow |C - D| + |D - C| + |C \cap D|$?
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- $|C - D| + |D - C| + |C \cap D|$: Actually, that should work. But, set difference can be tricky.
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- $|C| + |D| - |C \cap D|$?
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- $|C| + |D| - |C \cap D|$? Nailed it. Correct for double counting.
Inclusion-Exclusion rule for two sets

**Rule:** For two sets $S_1$ and $S_2$,

\[ |S_1 \cup S_2| = |S_1| + |S_2| - |S_1 \cap S_2|. \]
Inclusion-Exclusion rule for two sets

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Example:
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Example:

$S_1 = \{ \text{Tyler, Julie} \}$: HTAs with an $e$ in their name.
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- $S_1 \cap S_2 = \{ \text{Julie} \}$: HTAs with both a $j$ and an $e$ in their name.
- $S_1 \cup S_2 = \{ \text{Tyler, Julie, Julia} \}$: HTAs with either a $j$ or an $e$ in their name.
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- $S_1 \cup S_2 = \{ \text{Tyler, Julie, Julia} \}$: HTAs with either a $j$ or an $e$ in their name.
- $|\{ \text{Tyler, Julie, Julia} \}| = |\{ \text{Tyler, Julie} \}| + |\{ \text{Julie, Julia} \}| - |\{ \text{Julie} \}|$
Generalize to three sets

\[ S: \text{Set of all students in CS22.} \]
Generalize to three sets

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Generalize to three sets

$S$: Set of all students in CS22.

$D \subseteq S$: Set of all students in CS22 who have a pet dog.

$C \subseteq S$: Set of all students in CS22 who have a pet cat.

$B \subseteq S$: Set of all students in CS22 who have a pet bunny.
Generalize to three sets

$S$: Set of all students in CS22.

$D \subseteq S$: Set of all students in CS22 who have a pet dog.

$C \subseteq S$: Set of all students in CS22 who have a pet cat.

$B \subseteq S$: Set of all students in CS22 who have a pet bunny.

How express $|B \cup C \cup D|$ in terms of size of intersections of sets?
Visual analysis 1

\[ |B \cup C \cup D| = |B| + |C| + |D| + \ldots \]
Visual analysis 2

\[ |B \cup C \cup D| = |B| + |C| + |D| - |B \cap C| - |B \cap D| - |C \cap D| \ldots \]
Visual analysis 3

\[ |B \cup C \cup D| = |B| + |C| + |D| - |B \cap C| - |B \cap D| - |C \cap D| + |B \cap C \cap D| \]
Inclusion-Exclusion rule for three sets

**Rule:** For three sets $S_1$, $S_2$, $S_3$,

$$|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3|$$
$$- |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3|$$
$$+ |S_1 \cap S_2 \cap S_3|.$$
Inclusion-Exclusion rule for three sets

**Rule:** For three sets \( S_1, S_2, S_3, \)

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|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| \\
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+ |S_1 \cap S_2 \cap S_3|.
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**Example:**

\( S_1 = \{\text{Tyler, Julie}\} \): HTAs with an e in their name.
\( S_2 = \{\text{Julie, Julia}\} \): HTAs with an j in their name.
\( S_3 = \{\text{Tyler, Julie, Julia}\} \): HTAs with an l in their name.
\( S_1 \cap S_2 \cap S_3 = \{\text{Julie}\} \): HTAs with both a j and an e and an l in their name.

\[
|\{\text{Tyler, Julie, Julia}\}| = |\{\text{Tyler, Julie}\}| + |\{\text{Julie, Julia}\}| + |\{\text{Tyler, Julie, Julia}\}| - |\{\text{Julie}\}| - |\{\text{Tyler, Julie}\}| - |\{\text{Julie, Julia}\}| + |\{\text{Julie}\}|.
\]
Inclusion-Exclusion rule for three sets

**Rule:** For three sets $S_1$, $S_2$, $S_3$,

$$|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| + |S_1 \cap S_2 \cap S_3|.$$ 

Example:

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Inclusion-Exclusion rule for three sets

**Rule:** For three sets $S_1$, $S_2$, $S_3$,

$$|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3|$$

$$-|S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3|$$

$$+|S_1 \cap S_2 \cap S_3|.$$  

Example:

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$$+ |S_1 \cap S_2 \cap S_3|.$$ 

**Example:**

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- $S_3 = \{\text{Tyler, Julie, Julia}\}$: HTAs with an l in their name.
Inclusion-Exclusion rule for three sets

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$$|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| + |S_1 \cap S_2 \cap S_3|.$$

**Example:**

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- $S_1 \cap S_2 \cap S_3 = \{ \text{Julie} \}$: HTAs with both a $j$ and an $e$ and an $l$ in their name.
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**Rule:** For three sets $S_1$, $S_2$, $S_3$,

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|S_1 \cup S_2 \cup S_3| = |S_1| + |S_2| + |S_3| - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| + |S_1 \cap S_2 \cap S_3|.
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- $|\{ \text{Tyler, Julie, Julia} \}| = |\{ \text{Tyler, Julie} \}| + |\{ \text{Julie, Julia} \}| + |\{ \text{Tyler, Julie, Julia} \}| - |\{ \text{Julie} \}| - |\{ \text{Tyler, Julie} \}| - |\{ \text{Julie, Julia} \}| + |\{ \text{Julie} \}|$
Sets of permutations

In how many permutations of the set \(\{0, 1, 2, \ldots, 9\}\) do either 4 and 2, 0 and 4, or 6 and 0 appear consecutively?

\[
\frac{57}{104}
\]
Sets of permutations

In how many permutations of the set \(\{0, 1, 2, \ldots, 9\}\) do either 4 and 2, 0 and 4, or 6 and 0 appear consecutively?

Which of these permutations has this property?

▶ (4, 6, 5, 0, 1, 8, 3, 2, 9, 7)

\[ \frac{58}{104} \]
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- (4, 6, 5, 0, 1, 8, 3, 2, 9, 7) nope.
- (0, 4, 6, 1, 8, 5, 9, 3, 7, 2)
Sets of permutations

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Which of these permutations has this property?

- (4, 6, 5, 0, 1, 8, 3, 2, 9, 7) nope.
- (0, 4, 6, 1, 8, 5, 9, 3, 7, 2) 04!

\[
\frac{2}{5} \times 10! = 1451520.
\]

\(P_{60}\): permutations of 0 through 9 that contain 60.
\(P_{04}\): permutations of 0 through 9 that contain 04.
\(P_{42}\): permutations of 0 through 9 that contain 42.

Want:
\[
\left| P_{60} \cup P_{04} \cup P_{42} \right|.
\]
Sets of permutations

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- \((0, 4, 6, 1, 8, 5, 9, 3, 7, 2)\) 04!
- \((3, 4, 2, 0, 5, 6, 1, 9, 8, 7)\)
Sets of permutations

In how many permutations of the set \( \{0, 1, 2, \ldots, 9\} \) do either 4 and 2, 0 and 4, or 6 and 0 appear consecutively?

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- (3, 4, 2, 0, 5, 6, 1, 9, 8, 7) 42!
Sets of permutations

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- (3, 9, 4, 1, 2, 7, 0, 5, 6, 8)
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- \((3, 9, 4, 1, 2, 7, 0, 5, 6, 8)\) nope.
- \((0, 2, 6, 3, 7, 8, 4, 9, 5, 1)\)
Sets of permutations

In how many permutations of the set \( \{0, 1, 2, \ldots, 9\} \) do either 4 and 2, 0 and 4, or 6 and 0 appear consecutively?

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\[ \frac{2}{5} \times 10! = 1451520. \]
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In how many permutations of the set \( \{0, 1, 2, \ldots, 9\} \) do either 4 and 2, 0 and 4, or 6 and 0 appear consecutively?

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\[ \begin{align*}
&\quad (4, 6, 5, 0, 1, 8, 3, 2, 9, 7) \text{ nope.} \\
&\quad (0, 4, 6, 1, 8, 5, 9, 3, 7, 2) \text{ 04!} \\
&\quad (3, 4, 2, 0, 5, 6, 1, 9, 8, 7) \text{ 42!} \\
&\quad (3, 9, 4, 1, 2, 7, 0, 5, 6, 8) \text{ nope.} \\
&\quad (0, 2, 6, 3, 7, 8, 4, 9, 5, 1) \text{ nope.}
\end{align*} \]

\[ \frac{2}{5} \times 10! = 1451520. \]

\( P_{60} \): permutations of 0 through 9 that contain 60.
Sets of permutations

In how many permutations of the set \{0, 1, 2, \ldots, 9\} do either 4 and 2, 0 and 4, or 6 and 0 appear consecutively?

Which of these permutations has this property?

- (4, 6, 5, 0, 1, 8, 3, 2, 9, 7) nope.
- (0, 4, 6, 1, 8, 5, 9, 3, 7, 2) 04!
- (3, 4, 2, 0, 5, 6, 1, 9, 8, 7) 42!
- (3, 9, 4, 1, 2, 7, 0, 5, 6, 8) nope.
- (0, 2, 6, 3, 7, 8, 4, 9, 5, 1) nope.

\[\frac{2}{5} \times 10! = 1451520.\]

\(P_{60}\): permutations of 0 through 9 that contain 60.

\(P_{04}\): permutations of 0 through 9 that contain 04.
Sets of permutations

In how many permutations of the set \( \{0, 1, 2, \ldots, 9\} \) do either 4 and 2, 0 and 4, or 6 and 0 appear consecutively?

Which of these permutations has this property?

- (4, 6, 5, 0, 1, 8, 3, 2, 9, 7) nope.
- (0, 4, 6, 1, 8, 5, 9, 3, 7, 2) 04!
- (3, 4, 2, 0, 5, 6, 1, 9, 8, 7) 42!
- (3, 9, 4, 1, 2, 7, 0, 5, 6, 8) nope.
- (0, 2, 6, 3, 7, 8, 4, 9, 5, 1) nope.

\[
\frac{2}{5} \times 10! = 1451520.
\]

\( P_{60} \): permutations of 0 through 9 that contain 60.

\( P_{04} \): permutations of 0 through 9 that contain 04.

\( P_{42} \): permutations of 0 through 9 that contain 42.
Sets of permutations

In how many permutations of the set \{0, 1, 2, \ldots, 9\} do either 4 and 2, 0 and 4, or 6 and 0 appear consecutively?

Which of these permutations has this property?

- (4, 6, 5, 0, 1, 8, 3, 2, 9, 7) nope.
- (0, 4, 6, 1, 8, 5, 9, 3, 7, 2) 04!
- (3, 4, 2, 0, 5, 6, 1, 9, 8, 7) 42!
- (3, 9, 4, 1, 2, 7, 0, 5, 6, 8) nope.
- (0, 2, 6, 3, 7, 8, 4, 9, 5, 1) nope.

\[
\frac{2}{5} \times 10! = 1451520.
\]

\(P_{60}\): permutations of 0 through 9 that contain 60.

\(P_{04}\): permutations of 0 through 9 that contain 04.

\(P_{42}\): permutations of 0 through 9 that contain 42.

Want: \(|P_{60} \cup P_{04} \cup P_{42}|\).
Inclusion-exclusion, constrained permutation

\[ |P_{60} \cup P_{04} \cup P_{42}| \]
Inclusion-exclusion, constrained permutation

\[ |P_{60} \cup P_{04} \cup P_{42}| \]
\[ = |P_{60}| + |P_{04}| + |P_{42}| \]
\[ - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \]
\[ + |P_{60} \cap P_{04} \cap P_{42}| \]
Inclusion-exclusion, constrained permutation

\[ |P_{60} \cup P_{04} \cup P_{42}| = |P_{60}| + |P_{04}| + |P_{42}| - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| + |P_{60} \cap P_{04} \cap P_{42}| \]

\[ |P_{60}| = ? \]
Inclusion-exclusion, constrained permutation

\[ |P_{60} \cup P_{04} \cup P_{42}| \]
\[ = |P_{60}| + |P_{04}| + |P_{42}| \]
\[ - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \]
\[ + |P_{60} \cap P_{04} \cap P_{42}| \]
\[ |P_{60}| = ? \]

Clever trick: In \( P_{60} \), can view "60" as a unit.
Inclusion-exclusion, constrained permutation

\[ |P_{60} \cup P_{04} \cup P_{42}| \]
\[ = |P_{60}| + |P_{04}| + |P_{42}| \]
\[ - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \]
\[ + |P_{60} \cap P_{04} \cap P_{42}| \]

\[ |P_{60}| =? \]

Clever trick: In \( P_{60} \), can view “60” as a unit. So, each element of \( P_{60} \) is a permutation of \{1, 2, 3, 4, 5, 7, 8, 9, 60\}. 
Inclusion-exclusion, constrained permutation

\[ |P_{60} \cup P_{04} \cup P_{42}| \]
\[ = |P_{60}| + |P_{04}| + |P_{42}| \]
\[ - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \]
\[ + |P_{60} \cap P_{04} \cap P_{42}| \]

\[ |P_{60}| = ? \]

Clever trick: In \( P_{60} \), can view “60” as a unit. So, each element of \( P_{60} \) is a permutation of \( \{1, 2, 3, 4, 5, 7, 8, 9, 60\} \). Therefore, \( |P_{60}| = \)
Inclusion-exclusion, constrained permutation

\[ |P_{60} \cup P_{04} \cup P_{42}| \]
\[ = |P_{60}| + |P_{04}| + |P_{42}| - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| + |P_{60} \cap P_{04} \cap P_{42}| \]

\[ |P_{60}| = ? \]

Clever trick: In \( P_{60} \), can view “60” as a unit. So, each element of \( P_{60} \) is a permutation of \( \{1, 2, 3, 4, 5, 7, 8, 9, 60\} \). Therefore, \( |P_{60}| = 9! \).
Inclusion-exclusion, constrained permutation

\[ |P_{60} \cup P_{04} \cup P_{42}| \]
\[ = |P_{60}| + |P_{04}| + |P_{42}| \]
\[ - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \]
\[ + |P_{60} \cap P_{04} \cap P_{42}| \]

\[ |P_{60}| =? \]

Clever trick: In \( P_{60} \), can view “60” as a unit. So, each element of \( P_{60} \) is a permutation of \( \{1, 2, 3, 4, 5, 7, 8, 9, 60\} \). Therefore, \( |P_{60}| = 9! \).  

\[ |P_{04}| = \]
Inclusion-exclusion, constrained permutation

\[ |P_{60} \cup P_{04} \cup P_{42}| = |P_{60}| + |P_{04}| + |P_{42}| - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| + |P_{60} \cap P_{04} \cap P_{42}| \]

\[ |P_{60}| =? \]

Clever trick: In \( P_{60} \), can view “60” as a unit. So, each element of \( P_{60} \) is a permutation of \( \{1, 2, 3, 4, 5, 7, 8, 9, 60\} \). Therefore, \( |P_{60}| = 9! \). \( |P_{04}| = 9! \).
Inclusion-exclusion, constrained permutation

\[ |P_{60} \cup P_{04} \cup P_{42}| = |P_{60}| + |P_{04}| + |P_{42}| - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| + |P_{60} \cap P_{04} \cap P_{42}| \]

\[ |P_{60}| =? \]

Clever trick: In \( P_{60} \), can view “60” as a unit. So, each element of \( P_{60} \) is a permutation of \( \{1, 2, 3, 4, 5, 7, 8, 9, 60\} \). Therefore, \( |P_{60}| = 9! \). \( |P_{04}| = 9! \). \( |P_{42}| = \)
Inclusion-exclusion, constrained permutation

\[ |P_{60} \cup P_{04} \cup P_{42}| = |P_{60}| + |P_{04}| + |P_{42}| - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| + |P_{60} \cap P_{04} \cap P_{42}| \]

\[ |P_{60}| = ? \]

Clever trick: In \( P_{60} \), can view “60” as a unit. So, each element of \( P_{60} \) is a permutation of \( \{1, 2, 3, 4, 5, 7, 8, 9, 60\} \). Therefore, \( |P_{60}| = 9! \). \( |P_{04}| = 9! \). \( |P_{42}| = 9! \).
Pairwise intersections

\[
\begin{align*}
|P_{60} \cup P_{04} \cup P_{42}| &= |P_{60}| + |P_{04}| + |P_{42}| \\
&- |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \\
&+ |P_{60} \cap P_{04} \cap P_{42}|
\end{align*}
\]

\[= 3 \times 9!\]

\[-|P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \\
+ |P_{60} \cap P_{04} \cap P_{42}|
\]

\[|P_{60} \cap P_{04}| = ?\]
Pairwise intersections

\[
|P_{60} \cup P_{04} \cup P_{42}| \\
= |P_{60}| + |P_{04}| + |P_{42}| \\
- |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \\
+ |P_{60} \cap P_{04} \cap P_{42}| \\
= 3 \times 9! \\
- |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \\
+ |P_{60} \cap P_{04} \cap P_{42}|
\]

\[|P_{60} \cap P_{04}| = ? \text{ Trick works again! Can view “604” as a unit.}\]
Pairwise intersections

\[ |P_{60} \cup P_{04} \cup P_{42}| \]
\[ = |P_{60}| + |P_{04}| + |P_{42}| \]
\[ - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \]
\[ + |P_{60} \cap P_{04} \cap P_{42}| \]
\[ = 3 \times 9! \]
\[ - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \]
\[ + |P_{60} \cap P_{04} \cap P_{42}| \]

\[ |P_{60} \cap P_{04}| = \text{? Trick works again! Can view “604” as a unit. So, each element is a permutation of \{1, 2, 3, 5, 7, 8, 9, 604\}.} \]
Pairwise intersections

\[
|P_{60} \cup P_{04} \cup P_{42}| \\
= |P_{60}| + |P_{04}| + |P_{42}| \\
- |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \\
+ |P_{60} \cap P_{04} \cap P_{42}|
\]

\[
= 3 \times 9!
\]

\[
- |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}|
\]

\[
+ |P_{60} \cap P_{04} \cap P_{42}|
\]

\[
|P_{60} \cap P_{04}| = \text{? Trick works again! Can view “604” as a unit. So, each element is a permutation of \{1, 2, 3, 5, 7, 8, 9, 604\}. Therefore,}
\]
Pairwise intersections

\[ |P_{60} \cup P_{04} \cup P_{42}| \]
\[ = |P_{60}| + |P_{04}| + |P_{42}| - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| + |P_{60} \cap P_{04} \cap P_{42}| \]
\[ = 3 \times 9! \]
\[ - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| + |P_{60} \cap P_{04} \cap P_{42}| \]

\[ |P_{60} \cap P_{04}| = ? \] Trick works again! Can view “604” as a unit. So, each element is a permutation of \{1, 2, 3, 5, 7, 8, 9, 604\}. Therefore, 8!. 


Pairwise intersections

\[
|P_{60} \cup P_{04} \cup P_{42}| = |P_{60}| + |P_{04}| + |P_{42}| - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}|
+ |P_{60} \cap P_{04} \cap P_{42}|
= 3 \times 9! 
- |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}|
+ |P_{60} \cap P_{04} \cap P_{42}|

|P_{60} \cap P_{04}| =? \text{ Trick works again! Can view “604” as a unit. So, each element is a permutation of \{1, 2, 3, 5, 7, 8, 9, 604\}. Therefore, 8!.}

|P_{42} \cap P_{04}| =?
Pairwise intersections

\[ |P_{60} \cup P_{04} \cup P_{42}| \]
\[ = |P_{60}| + |P_{04}| + |P_{42}| \]
\[ - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \]
\[ + |P_{60} \cap P_{04} \cap P_{42}| \]
\[ = 3 \times 9! \]
\[ - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \]
\[ + |P_{60} \cap P_{04} \cap P_{42}| \]

\[ |P_{60} \cap P_{04}| =? \] Trick works again! Can view “604” as a unit. So, each element is a permutation of \( \{1, 2, 3, 5, 7, 8, 9, 604\} \). Therefore, 8!.

\[ |P_{42} \cap P_{04}| =? \] Trick works again! Can view “042” as a unit. So,
Pairwise intersections

\[ |P_{60} \cup P_{04} \cup P_{42}| = |P_{60}| + |P_{04}| + |P_{42}| - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| + |P_{60} \cap P_{04} \cap P_{42}| \]

= 3 \times 9!

\[ |P_{60} \cap P_{04}| =? \text{ Trick works again! Can view “604” as a unit. So, each element is a permutation of \{1, 2, 3, 5, 7, 8, 9, 604\}. Therefore, 8!}. \]

\[ |P_{42} \cap P_{04}| =? \text{ Trick works again! Can view “042” as a unit. So, 8!}. \]
Pairwise intersections

$$|P_{60} \cup P_{04} \cup P_{42}|$$

$$= |P_{60}| + |P_{04}| + |P_{42}|$$

$$- |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}|$$

$$+ |P_{60} \cap P_{04} \cap P_{42}|$$

$$= 3 \times 9!$$

$$- |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}|$$

$$+ |P_{60} \cap P_{04} \cap P_{42}|$$

$|P_{60} \cap P_{04}| = ?$ Trick works again! Can view “604” as a unit. So, each element is a permutation of \{1, 2, 3, 5, 7, 8, 9, 604\}. Therefore, 8!.

$|P_{42} \cap P_{04}| = ?$ Trick works again! Can view “042” as a unit. So, 8!.

$|P_{60} \cap P_{42}| = ?$
Pairwise intersections

\[ |P_{60} \cup P_{04} \cup P_{42}| \]
\[ = |P_{60}| + |P_{04}| + |P_{42}| \]
\[ - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \]
\[ + |P_{60} \cap P_{04} \cap P_{42}| \]
\[ = 3 \times 9! \]
\[ - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \]
\[ + |P_{60} \cap P_{04} \cap P_{42}| \]

| \( P_{60} \cap P_{04} \) | =? Trick works again! Can view “604” as a unit. So, each element is a permutation of \( \{1, 2, 3, 5, 7, 8, 9, 604\} \). Therefore, 8!.

| \( P_{42} \cap P_{04} \) | =? Trick works again! Can view “042” as a unit. So, 8!.

| \( P_{60} \cap P_{42} \) | =? Trick fails!
Pairwise intersections

\[ |P_{60} \cup P_{04} \cup P_{42}| = |P_{60}| + |P_{04}| + |P_{42}| - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| + |P_{60} \cap P_{04} \cap P_{42}| \]

= 3 \times 9!

\[ -|P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| + |P_{60} \cap P_{04} \cap P_{42}| \]

\[ |P_{60} \cap P_{04}| = ? \] Trick works again! Can view “604” as a unit. So, each element is a permutation of \{1, 2, 3, 5, 7, 8, 9, 604\}. Therefore, 8!.

\[ |P_{42} \cap P_{04}| = ? \] Trick works again! Can view “042” as a unit. So, 8!.

\[ |P_{60} \cap P_{42}| = ? \] Trick fails! Wait, no, just changes.
Pairwise intersections

\[ |P_{60} \cup P_{04} \cup P_{42}| = |P_{60}| + |P_{04}| + |P_{42}| \]
\[ - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \]
\[ + |P_{60} \cap P_{04} \cap P_{42}| \]
\[ = 3 \times 9! \]

\[ |P_{60} \cap P_{04}| = ? \text{ Trick works again! Can view “604” as a unit. So, each element is a permutation of } \{1, 2, 3, 5, 7, 8, 9, 604\}. \text{ Therefore, } 8!. \]

\[ |P_{42} \cap P_{04}| = ? \text{ Trick works again! Can view “042” as a unit. So, } 8!. \]

\[ |P_{60} \cap P_{42}| = ? \text{ Trick fails! Wait, no, just changes. Now, each element is a permutation of } \{1, 3, 5, 7, 8, 9, 60, 42\}. \]
Pairwise intersections

$$|P_{60} \cup P_{04} \cup P_{42}| = |P_{60}| + |P_{04}| + |P_{42}|$$

$$-|P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}|$$

$$+|P_{60} \cap P_{04} \cap P_{42}|$$

$$= 3 \times 9!$$

$$-|P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}|$$

$$+|P_{60} \cap P_{04} \cap P_{42}|$$

$$|P_{60} \cap P_{04}| =? \text{ Trick works again! Can view “604” as a unit. So, each element is a permutation of } \{1, 2, 3, 5, 7, 8, 9, 604\}. \text{ Therefore, 8!}.$$ 

$$|P_{42} \cap P_{04}| =? \text{ Trick works again! Can view “042” as a unit. So, 8!}.$$ 

$$|P_{60} \cap P_{42}| =? \text{ Trick fails! Wait, no, just changes. Now, each element is a permutation of } \{1, 3, 5, 7, 8, 9, 60, 42\}. \text{ Still 8!}.$$
Three-way intersection

$$|P_{60} \cup P_{04} \cup P_{42}|$$

$$= |P_{60}| + |P_{04}| + |P_{42}|$$

$$- |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}|$$

$$+ |P_{60} \cap P_{04} \cap P_{42}|$$

$$= 3 \times 9! - 3 \times 8!$$

$$+ |P_{60} \cap P_{04} \cap P_{42}|$$

$$|P_{60} \cap P_{04} \cap P_{42}| = ?.$$
Three-way intersection

\[
|P_{60} \cup P_{04} \cup P_{42}| \\
= |P_{60}| + |P_{04}| + |P_{42}| \\
- |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \\
+ |P_{60} \cap P_{04} \cap P_{42}| \\
= 3 \times 9! - 3 \times 8! + |P_{60} \cap P_{04} \cap P_{42}|
\]

\[
|P_{60} \cap P_{04} \cap P_{42}| =? . \text{ Yay, trick works again! Can view “6042” as a unit.}
\]
Three-way intersection

\[ |P_{60} \cup P_{04} \cup P_{42}| \]
\[ = |P_{60}| + |P_{04}| + |P_{42}| - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| + |P_{60} \cap P_{04} \cap P_{42}| \]
\[ = 3 \times 9! - 3 \times 8! + |P_{60} \cap P_{04} \cap P_{42}| \]

\[ |P_{60} \cap P_{04} \cap P_{42}| = ? \]  
Yay, trick works again! Can view “6042” as a unit. So, each element is a permutation of \{1, 3, 5, 7, 8, 9, 6042\}. 
Three-way intersection

\[ |P_{60} \cup P_{04} \cup P_{42}| = |P_{60}| + |P_{04}| + |P_{42}| - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| + |P_{60} \cap P_{04} \cap P_{42}| \]

\[ = 3 \times 9! - 3 \times 8! + |P_{60} \cap P_{04} \cap P_{42}| \]

\[ |P_{60} \cap P_{04} \cap P_{42}| = ?. \text{ Yay, trick works again! Can view “6042” as a unit. So, each element is a permutation of \{1, 3, 5, 7, 8, 9, 6042\}. Therefore,} \]
Three-way intersection

\[ |P_{60} \cup P_{04} \cup P_{42}| \]
\[ = |P_{60}| + |P_{04}| + |P_{42}| \]
\[ - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \]
\[ + |P_{60} \cap P_{04} \cap P_{42}| \]
\[ = 3 \times 9! - 3 \times 8! + 7! \]

\[ |P_{60} \cap P_{04} \cap P_{42}| = ? \]. Yay, trick works again! Can view “6042” as a unit. So, each element is a permutation of \{1, 3, 5, 7, 8, 9, 6042\}. Therefore, 7!. 

Sampling estimate was actually pretty good. Guessed 20% of permutations, but it’s closer to 27%. 

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Three-way intersection

\[ |P_{60} \cup P_{04} \cup P_{42}| \]
\[ = |P_{60}| + |P_{04}| + |P_{42}| \]
\[ - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \]
\[ + |P_{60} \cap P_{04} \cap P_{42}| \]
\[ = 3 \times 9! - 3 \times 8! + 7! \]

\[ |P_{60} \cap P_{04} \cap P_{42}| = ?. \text{ Yay, trick works again! Can view “6042” as a unit. So, each element is a permutation of \{1, 3, 5, 7, 8, 9, 6042\}. Therefore, 7!.} \]

\[ |P_{60} \cup P_{04} \cup P_{42}| = 3 \times 9! - 3 \times 8! + 7! = 972720. \]
Three-way intersection

\[ |P_{60} \cup P_{04} \cup P_{42}| \]
\[ = |P_{60}| + |P_{04}| + |P_{42}| \]
\[ - |P_{60} \cap P_{04}| - |P_{42} \cap P_{04}| - |P_{60} \cap P_{42}| \]
\[ + |P_{60} \cap P_{04} \cap P_{42}| \]
\[ = 3 \times 9! - 3 \times 8! + 7! \]

\[ |P_{60} \cap P_{04} \cap P_{42}| = ? \]. Yay, trick works again! Can view “6042” as a unit. So, each element is a permutation of \{1, 3, 5, 7, 8, 9, 6042\}. Therefore, 7!.

\[ |P_{60} \cup P_{04} \cup P_{42}| = 3 \times 9! - 3 \times 8! + 7! = 972720. \]

Sampling estimate was actually pretty good. Guessed 20% of permutations, but it’s closer to 27%.
n-way Inclusion-Exclusion

\[ \left| S_1 \cup S_2 \cup \cdots \cup S_n \right| = \]

the sum of the sizes of the individual sets
minus the sizes of all two-way intersections
plus the sizes of all three-way intersections
minus the sizes of all four-way intersections
plus the sizes of all five-way intersections, etc.

Hyper-mathy version:

\[ \left| \bigcup_{i=1}^{n} S_i \right| = \sum_{X \in \mathcal{P}([1,n]) - \emptyset} (-1)^{|X|+1} \left| \bigcap_{i \in X} S_i \right| \]