Functions, Injectivity, Surjectivity, Bijections

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Overview

Relation Diagrams (4.4.1)

Relational Images (4.4.2)
Binary relations

Definition. A binary relation, $R$, consists of a set, $A$, called the domain of $R$, a set, $B$, called the codomain of $R$, and a subset of $A \times B$ called the graph of $R$. 

![Binary relation diagram]
Properties of relations

A binary relation:

- is a *partial function* when it has the \([\leq 1 \text{ arrow out}]\) property.
  Book: “function”. Us: ”function” is \([= 1 \text{ arrow out}]\) property.
- is *surjective* when it has the \([\geq 1 \text{ arrows in}]\) property.
- is *total* when it has the \([\geq 1 \text{ arrows out}]\) property.
- is *injective* when it has the \([\leq 1 \text{ arrow in}]\) property.
- is *bijective* when it has both the \([= 1 \text{ arrow out}]\) and the \([= 1 \text{ arrow in}]\) properties.
Example relation #1

partial function: \([\leq 1 \text{ out}]\). surjective: \([\geq 1 \text{ in}]\). total: \([\geq 1 \text{ out}]\).
injective: \([\leq 1 \text{ in}]\). bijective: \([= 1 \text{ out}]\) and \([= 1 \text{ in}]\).

total, surjective, function.
Example relation #2

partial function: $\leq 1$ out. surjective: $\geq 1$ in. total: $\geq 1$ out. injective: $\leq 1$ in. bijective: $= 1$ out and $= 1$ in.

total, injective, function.
Example relation #3

partial function: $\leq 1$ out]. surjective: $\geq 1$ in]. total: $\geq 1$ out].
injective: $\leq 1$ in]. bijective: $= 1$ out] and $= 1$ in].

Equation $y = 1/x^2$ on $\mathbb{R}^+$. $x$ is an element in the domain, $y$ is an element in the co-domain.

bijective partial function. (implies surjective, total, injective.)
Example relation #4

partial function: \([\leq 1 \text{ out}].\) surjective: \([\geq 1 \text{ in}].\) total: \([\geq 1 \text{ out}].\)
injective: \([\leq 1 \text{ in}].\) bijective: \([= 1 \text{ out}]\) and \([= 1 \text{ in}].\)

Equation \(y = 1/x^2\) on \(\mathbb{R} \)
Image definition

Definition. The image of a set $Y$ under a relation $R$, written $R(Y)$, is the subset of elements of the codomain $B$ of $R$ that are related to some element in $Y$.

In terms of the relation diagram, $R(Y)$ is the set of points with an arrow coming in that starts from some point in $Y$.

$$R(Y) = \{x \in B \mid \exists y \in Y, y \ R \ x\}.$$
Inverse definition

Definition: The *inverse* $R^{-1}$ of a relation $R : A \rightarrow B$ is the relation from $B$ to $A$ defined by the rule

$$b R^{-1} a \iff a R b.$$  

Definition: The image of a set under the relation $R^{-1}$ is called the *inverse image* of the set. That is, the inverse image of a set $X$ under the relation $R$ is defined to be $R^{-1}(X)$.

Example: $x R y$ iff there's a word with first letter $x$ and second letter $y$. The image $R(\{c, k\})$ is the letters that can appear after $c$ or $k$ at the beginning of a word. It's the set $\{a, b, e, h, i, l, n, o, r, s, t, u, v, w, y, z\}$.

The inverse image $R^{-1}(\{c, k\})$ is the letters that can appear before $c$ or $k$ at the beginning of a word. It's the set $\{a, e, i, o, s, t, u, y\}$.
Inverses of relations

What can we infer about $R^{-1}$ if $R$ is:

- partial function? injective
- surjective? total
- total? surjective
- injective? partial function
- bijective? bijective
- function? injective and surjective
More examples to consider

Make natural examples for each combination of properties.

- $\sqrt{x}$ on $\mathbb{R}$
- $\sqrt{16 - \sqrt{x}}$ on $\mathbb{R}$
- $|x + 10|$ on $\mathbb{Z}$
- $|x \mod 2|$ on $\mathbb{Z}$
- $\sin(x)$ on $\mathbb{R}$