Strong Induction

Requirements

1. Formally **define the predicate** that will be proved inductively.

2. Prove that the predicate holds in the **base case**.

3. Formally state the **inductive hypothesis**.

4. Assume the inductive hypothesis, and prove the **inductive step**.

5. **Conclude** that the predicate holds in general.

Example

Prove that every integer \( n \geq 2 \) can be written as a product of one or more prime numbers.

Proof

Let \( P(n) \) be the predicate “\( n \) can be written as a product of one or more prime numbers”.

**Base case.** The integer 2 is prime, so it is a product of exactly one prime number (itself). Therefore, \( P(2) \) is true.

**Inductive Hypothesis.** Assume the inductive hypothesis, that for a particular \( k \), \( P(i) \) is true for all \( 2 \leq i \leq k \).

**Inductive Step.** We must prove \( P(k+1) \), that \( k + 1 \) is the product of one or more prime numbers. \( k + 1 \) is either prime or composite. If it is prime, then it is the product of exactly one prime number (itself), and \( P(k + 1) \) is true. If it is composite, then by definition it is the product of two factors, \( k + 1 = ab \), where \( a \) and \( b \) are integers \( \geq 2 \). Since \( a \) and \( b \) are both greater than 1, they must also both be less than \( k + 1 \). By the inductive hypothesis, \( a \) and \( b \) can each be written as a product of one or more primes. But since \( k + 1 = ab \), we can combine these two products to express \( k + 1 \) as a product of primes, so \( P(k + 1) \) is true.

**Conclusion.** Since \( P(2) \) is true and \( P(2), \ldots, P(k) \) together imply \( P(k + 1) \), \( P(n) \) is true for all integers \( n \geq 2 \). \( \square \)