Division into Cases

Requirements

1. Show that the proposition always falls into one of a few cases.
2. List the cases.
3. Under each case, give a proof that the proposition holds for that case.
4. Conclude that the overall proposition holds.

Example

Prove that the square of any odd integer has the form $8m + 1$ for some integer $m$.

Proof:

Suppose $n$ is an odd integer. By the Quotient-Remainder Theorem, $n$ can be written as $4q + r$, where $q$ and $r$ are integers and $0 \leq r < 4$. Because $4q$ and $4q + 2$ are even, $n$ must be of the form $4q + 1$ or $4q + 3$.

Case 1: $n = 4q + 1$.

Proof of Case 1:

\[
\begin{align*}
    n^2 &= (4q + 1)^2 \\
         &= 16q^2 + 8q + 1 \\
         &= 8(2q^2 + q) + 1
\end{align*}
\]

Let $m = 2q^2 + q$. $m$ is an integer, because 2 and $q$ are integers and the sums and products of integers are integers. Substituting, we get $n^2 = 8m + 1$ where $m$ is an integer.

Case 2: $n = 4q + 3$.

Proof of Case 2:

\[
\begin{align*}
    n^2 &= (4q + 3)^2 \\
         &= 16q^2 + 24q + 9 \\
         &= 8(2q^2 + 4q + 1) + 1
\end{align*}
\]

Let $m = 2q^2 + 4q + 1$. $m$ is an integer, because it is the sum of products of integers. Substituting, we get $n^2 = 8m + 1$ where $m$ is an integer.

Cases 1 and 2 show that for any odd integer $n$, $n^2 = 8m + 1$ where $m$ is an integer. This completes the proof. \[\square\]