Final Practice Problems

Due: Never

All homeworks are due at 12:55 PM on Gradescope.

Please do not include any identifying information about yourself in the handin, including your Banner ID.

Be sure to fully explain your reasoning and show all work for full credit.

Problem 1

RISD, Newport, and Narragansett beaches each have exactly 4 seashells, as follows:

- RISD Beach has 3 conch shells and 1 sand dollar.
- Newport has 2 conch shells and 2 sand dollars.
- Narragansett has 4 conch shells.

You and your friend Donny are indecisive and choose one of these beaches to visit uniformly at random. At this beach, you practice your favorite hobby: picking up seashells uniformly at random. Donny picks up a shell first, and you observe that he got a conch. What is the probability that you will pick up a sand dollar?

Hint: First find the probability of going to RISD Beach, Newport, or Narragansett, given that Donny got a conch.

See xkcd comic 1236 for more fun seashell probability.

Problem 2

A complete bipartite graph is a graph \(^1\) whose vertices can be partitioned into two sets, \(L\) and \(R\) such that the following conditions hold: every edge has one endpoint in \(L\) and the other in \(R\), and every vertex in \(L\) is connected to every vertex in \(R\) via an edge and vice versa. See the figure below for examples.

---

\(^1\)An equally important (non-graph theory related) graph: https://xkcd.com/815/
In this problem, we’ll use the notation $K_{m,n}$ to refer to the complete bipartite graph with disjoint vertex sets, $M$ and $N$, with $|M| = m$ and $|N| = n$ respectively.

Find the expected distance between two distinct, randomly selected nodes in the complete bipartite graph $K_{m,n}$. Do not simplify your answer.

Problem 3

Let $S$ be a subset of a universal set $U$. The *indicator function* (or characteristic function) $f_S$ of $S$ indicates whether or not a given element from $U$ is a member of $S$. Formally, $f_S$ is defined as the function from $U$ to $\{0, 1\}$ such that $f_S(x) = 1$ if $x \in S$ and $f_S(x) = 0$ otherwise.

Show the following holds for all $x$, $A$, and $B$, where $x$ is some member of $U$, and $A$ and $B$ are some subsets of $U$:

a. $f_{A \cap B}(x) = f_A(x)f_B(x)$.

b. $f_{A \cup B}(x) = f_A(x) + f_B(x) - f_A(x)f_B(x)$.

c. $f_{\overline{A}}(x) = 1 - f_A(x)$.

d. $f_{A \setminus B}(x) = f_A(x)(1 - f_B(x))$.

Problem 4

At the national gathering of the Donut Club, there are $n$ people.

A donut enthusiast at the gathering is popular if they have strictly more friends than the average number of friends their friends have. Note that friendship is bidirectional; that is, if Person A is friends with Person B, Person B is friends with Person A.
We can model this problem as a graph, where each vertex represents a person and undirected edges represent friendships.

a. Prove that there exists a graph $G = (V, E)$ with $n - 2$ popular people for all $n \geq 3$.

b. Prove, using a contradiction, that for any graph $G$, not all $n$ people can be popular.

**Hint:** Consider this definition of popularity. A vertex $v$ is popular if and only if:

$$\deg(v) > \sum_{u \in N(v)} \frac{\deg(u)}{\deg(v)}$$

where $N(v)$ is the neighbors of $v$. Show that every person being popular would lead to a contradiction.

c. Prove that on average, people have the same or fewer friends than their friends do. That is, the average number of friends a person has, summing over all people, is less than or equal to the average number of friends each of that person’s friends has.

**Hint:** Think about the difference between picking a random *person* in the entire graph and picking a random *friend* (not of anyone in particular, but just of anyone). Picking a random *friend* biases the sample towards people who are more popular.

**Problem 5**

Given an arrangement of $n$ red dots and $n$ blue dots in a circle, prove that there exists a starting point such that if you go around the circle clockwise you will always see at least as many red dots as blue dots. **Hint:** You will need to use build down induction. Assume your inductive hypothesis holds for a circle of $2k$ total dots. Now consider an arbitrary circle of $2(k + 1)$ total dots, is there any way to build down?

**Problem 6**

Let $p_1$, $p_2$, and $p_3$ be distinct primes and let $r_1$, $r_2$, and $r_3$ be positive integers.

a. Let $S = p_1^{r_1} p_2^{r_2}$.

   i. How many integers between 1 and $S$ (inclusive) are divisible by $p_1$?
   
   ii. How many integers between 1 and $S$ (inclusive) are divisible by $p_2$?
   
   iii. How many are divisible by both $p_1$ and $p_2$?
iv. "The number of integers that are not relatively prime to $S$ is simply equal to the sum of the number of integers that are not divisible by $p_1$ and the number of integers divisible by $p_2$.

What is wrong with this logic? Find the actual number of integers between 1 and $S$ that are not relatively prime to $S$.

b. Let $S' = p_1^{r_1} p_2^{r_2} p_3^{r_3}$. How many integers between 1 and $S'$ are relatively prime to $S'$?

**Problem 7**

So far in this class, we have only looked at binary logic (everything can be represented as True or False). This is just a subset of a larger system of $n$-value logic systems, where there are $n$ possible values that can be used. In this problem, we will look at 3-value logic, also called ternary logic.

**Note:** This problem is the only place where we use ternary logic. Other problems on homeworks or exams will continue to just use two value logic unless explicitly stated otherwise.

In the 3-value logic system that we’ll use, a third value is introduced: Unknown. In an abstract sense, this represents cases where it cannot be determined if something is true or false. The following truth table demonstrates how Unknown behaves with our usual logical operators:

<table>
<thead>
<tr>
<th>$p$</th>
<th>$q$</th>
<th>$p \implies q$</th>
<th>$p \land q$</th>
<th>$p \lor q$</th>
<th>$\neg p$</th>
<th>$?p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>U</td>
<td>U</td>
<td>U</td>
<td>T</td>
<td>F</td>
<td>U</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>U</td>
<td>T</td>
<td>F</td>
<td>U</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>U</td>
<td>T</td>
<td>T</td>
<td>U</td>
<td>T</td>
<td>U</td>
<td>T</td>
</tr>
<tr>
<td>U</td>
<td>F</td>
<td>U</td>
<td>F</td>
<td>U</td>
<td>U</td>
<td>T</td>
</tr>
<tr>
<td>U</td>
<td>U</td>
<td>U</td>
<td>U</td>
<td>U</td>
<td>U</td>
<td>T</td>
</tr>
</tbody>
</table>

In addition to the usual operators, a useful operator for working with Unknown is the $?$ operator, which returns true if a value is Unknown and false otherwise. Using gates that correspond to the operators given in the table above, construct circuits with the following criteria:

**Note:** For building circuits, you can only use AND, OR, NOT, or $?$ gates.
a. Construct a logic circuit that will return true if both inputs are unknowns, and false otherwise.
b. Construct a logic circuit where there are two known inputs that return true if the inputs are different, and false if they are the same.
c. Build a logic circuit where, given that there is always exactly one unknown input, the opposite of the non-unknown input is returned.
d. Combine the three circuits you constructed for parts a-c to create one master circuit with the following criteria:
   - If both inputs are unknown, return true.
   - If one input is unknown, return the opposite of the known input.
   - If both inputs are known, return false.

Problem 8

Prove that for any set of \( n \) positive integers, there exists a non-empty subset such that the sum of all elements in the subset, \( k \), is divisible by \( n \).

**Hint:** Call the \( n \) elements of \( S \) \( s_1, s_2, \ldots, s_n \). Consider the \( n \) sums:

\[
\begin{align*}
& s_1 \\
& s_1 + s_2 \\
& s_1 + s_2 + s_3 \\
& \quad \vdots \\
& s_1 + s_2 + s_3 + \ldots + s_n
\end{align*}
\]

Problem 9

Consider a relation \( R \) on a non-empty set \( S \) such that:

1. No element in \( S \) is related to itself. That is, for all \( x \in S \), \((x, x)\) is not in the relation.
2. All elements in \( S \) are related to something. That is, for all \( x \in S \), there exists a \( y \in S \) such that \((x, y)\) is in the relation.
3. Transitivity holds.

We say that \( x_1 R^i y \) exactly when \( x_1 \) can reach \( y \) in \( i \) “hops”. That is, when there exists some \( x_2, x_3, \ldots, x_i \) such that \((x_1, x_2), (x_2, x_3), \ldots, (x_{i-1}, x_i), (x_i, y) \in R \).
a. Prove that for all $y \in S$, for any $x_1 \in S$, $i \in \mathbb{N}$ such that $x_1 R^i y$, $y$ cannot be related to $x_1$

b. Using part a, prove that $S$ cannot be finite.

Problem 10

A primitive Pythagorean triple is an ordered triple of positive integers $(a, b, c)$ such that $a^2 + b^2 = c^2$ and $a, b, c$ have no common divisors. For example, $(3, 4, 5)$ is a primitive Pythagorean triple, but $(6, 8, 10)$ is not.

We’ll prove that for any primitive Pythagorean triple, exactly one of $a$ or $b$ is a multiple of 3.

a. Consider a general primitive Pythagorean triple, $(a, b, c)$, where $a^2 + b^2 = c^2$. Show that $a$ and $b$ can’t both be multiples of 3.

b. Next, prove that either $a$ or $b$ must be a multiple of 3.