Introduction to SET

The family card game SET uses a deck of 81 unique cards, each showing a different configuration of colored symbols. Each card is has a unique combination of one of each trait listed below:

<table>
<thead>
<tr>
<th>Count</th>
<th>Shape</th>
<th>Color</th>
<th>Shading</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Oval</td>
<td>Purple</td>
<td>Solid</td>
</tr>
<tr>
<td>2</td>
<td>Diamond</td>
<td>Red</td>
<td>Striped</td>
</tr>
<tr>
<td>3</td>
<td>Squiggle</td>
<td>Green</td>
<td>Open</td>
</tr>
</tbody>
</table>

Example cards look like this:

(green, blue, red)

The gameplay consists of 12 cards being dealt from the deck onto a table. Players then race to identify a SET of three cards. SETs are defined as follows:

- They all have the same symbol count, or all have different counts.
- They all have the same shape, or all have different shapes
- They are all the same color, or are all different colors.
- They all have the same shading, or all have different shadings.

The three cards shown above constitute a SET, as do the following triplets:

(red, red, red)
(blue, green, red)
(green, blue, red)
When a player identifies a SET, they pick up those three cards, set them aside in their scoring pile, and replace them on the table with three more cards. The likelihood of there existing at least one obtainable set with 12 cards on the table is pretty high, but not 100%. If there does not exist one, then three more cards are added onto the table for a total of 15. While there still does not exist an obtainable SET, three more cards are continuously added until there exists one. Unfortunately, it is very computationally hard (beyond the scope of this course) to determine the chance of at least one SET occurring in some \( n \) cards. In this recitation, we will instead explore the chances of being able to make a SET in other cases, as well as other interesting combinatorial properties of the game.

**SET Counting**

Consider an arbitrary SET containing random cards \( C_1, C_2, C_3 \), each of which can take on a card value from 1 to 81, inclusive.

1. What is the probability that \( C_1, C_2, C_3 \) form a set given \( C_1 \) and \( C_2 \)? What is the probability that three randomly drawn cards form a set?

2. How many unique SETs can be created containing some card \( c_x \)? (i.e. how many different SETs contain a given card in the deck?)
3. How many unique SETs can be created in total from the 81 card deck?

4. What is the optimal strategy for searching for a SET? That is, should one start by looking for SETs with all 4 attributes different, with 3 attributes different and 1 the same, 2 and 2, or 1 and 3?
   **Hint:** Compute the probability that, given a group of three cards constitutes a SET, it falls under one of the above categories.

5. What is the expected number of SETs obtainable from 12 randomly drawn cards?
   **Hint:** Think about what random variable might be useful to define here, and then invoke linearity of expectation.
6. Prove that 5 cards that share two common fixed attributes (e.g. 5 cards that are all red and striped) must contain a SET.

7. Finally, prove that if the game has progressed to the point where only 3 card remain on the table (i.e. 26 SETs have been identified), those 3 cards constitute a SET.

**Hint:** Consider what happens to number of cards in the deck with each of the 3 versions of a given attribute whenever a SET is removed. Using modular arithmetic, you can then show that the last 3 cards MUST satisfy the rules for a SET for each of the 4 attributes.

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1'This doesn't always occur; games end when no more sets can be obtained at 6, 9, 12 or more cards