Recitation 8
Counting and Pascal’s Triangle

Review

Sort the following in order from largest to smallest. You may use a calculator or Wolfram Alpha.

i. The number of permutations of all the letters in the alphabet.

ii. The number of subsets of the letters of the alphabet of size 6 (one such subset is \{a, b, c, d, e, f\}).

iii. The number of 6 letter words made of unique letters (one such 6-letter word is “abcdef”).

iv. The number of subsets of the set of all the letters in the alphabet

v. The number of subsets of size 20 of the letters of the alphabet

Warm-Up

Use a counting argument to prove the following identity:

\[
C(n, k) = C(n-2, k-2) + 2C(n-2, k-1) + C(n-2, k)
\]
Fun with Pascal’s Triangle

Recall Pascal’s Triangle, where each term is found by summing the term to its upper left and the term to its upper right. The first 7 rows are depicted as follows.

\[
\begin{array}{ccccccc}
  n = 0 & 1 \\
  n = 1 & 1 & 1 \\
  n = 2 & 1 & 2 & 1 \\
  n = 3 & 1 & 3 & 3 & 1 \\
  n = 4 & 1 & 4 & 6 & 4 & 1 \\
  n = 5 & 1 & 5 & 10 & 10 & 5 & 1 \\
  n = 6 & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\
\end{array}
\]

We saw in class how these numbers relate to the binomial coefficients (namely the \(n^{th}\) row contains \(\binom{n}{0} ... \binom{n}{n}\)). However, there are lots of other cool patterns that arise from this triangle.

Part 1: Triangle Numbers

Triangle numbers are numbers that can be written as \(\sum_{i=1}^{k} i\). Find the sequence of the triangle numbers in Pascal’s Triangle. It may be helpful to write a few out first. Where does this sequence appear?
Prove that the part of Pascal’s Triangle that you have identified as the Triangle Sequence does, in fact, contain the triangle numbers (hint: Induction).

Prove that if you take any two consecutive triangle number, the result will be a square number (This proof can be done in many ways, so see if you can prove it in more than one way!)
Part 2: Powers of 11
Prove that if you concatenate the number in the \( n^{th} \) row of the sequence, then you will get \( 11^n \). Note: if a number is more than 1 digit, it is carried over to the next largest term. **Hint:** \( 11 = 10 + 1 \)

Part 3: Pascal Triangle Scavenger Hunt
Can you find the Fibonacci numbers (you may have to sum elements together/ look across multiple rows)?
**hint:** Write Pascals Triangle as a right triangle