Recitation 4
Induction and Algorithms

Warm-Up Challenge

a. Let $A$ be some arbitrary set. Consider the following set: $B = \{ x \in A \mid x \notin x \}$. Could $B$ be a member of $A$? [Hint: Suppose $B$ were a member of $A$. It must be that either $B$ is a member of $B$ or $B$ is not a member of $B$. Work out both cases.]

We say the set $X$ contains itself if $X$ itself is a member of $X$. We say a set $X$ does not contain itself if $X$ is not a member of $X$.

b. Call the set of all sets that don’t contain themselves, $S$. Can such an $S$ exist? Why or why not? [Hint: Suppose such an $S$ exists. $S$ must either contain or not contain itself. Work out both cases.]

Review

We will review the template for an inductive proof.

For example, say we are trying to prove that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$ is true for all $n \in \mathbb{N}$.

1. Define the predicate $P(n)$.

   Let $P(n)$ be the predicate that $\sum_{i=0}^{n} i = \frac{n(n+1)}{2}$.
2. Show that the base case is true.

   We will first show \( P(0) \) is true. \( \sum_{i=0}^{0} i = 0 \) and \( \frac{0(0+1)}{2} = 0 \) so they are equal as needed.

3. Assume the inductive hypothesis is true. If you are using standard induction then you will assume \( P(k) \) is true for some integer \( k \). If you are using strong induction then you will assume \( P(i) \) is true for all \( i \leq k \). Here, we are using standard induction.

   Assume \( P(k) \) is true for some integer \( k \).

4. Show that \( P(k + 1) \) is true given the inductive hypothesis.

   We will now show that \( \sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2} \).

   We know that \( \sum_{i=0}^{k+1} i = \left( \sum_{i=0}^{k} i \right) + (k + 1) \).

   By our inductive hypothesis \( \sum_{i=0}^{k} i = \frac{k(k+1)}{2} \).

   Therefore

   \[
   \begin{align*}
   \sum_{i=0}^{k+1} i &= \left( \sum_{i=0}^{k} i \right) + (k + 1) \\
   &= \frac{k(k+1)}{2} + (k + 1) \\
   &= \frac{k(k+1) + 2(k+1)}{2} \\
   &= \frac{(k + 1)(k + 2)}{2}
   \end{align*}
   \]

   as needed. \( \square \)
Warm-up

Prove by induction that

\[(1 - \frac{1}{4})(1 - \frac{1}{9})...(1 - \frac{1}{n^2}) = \frac{n + 1}{2n}\]

Checkpoint - Call a TA over
Section Lesson - Algorithms

Why does induction work?

You can think of induction as a ladder. We want to prove that we can reach every step of the ladder.

The base case says that we can reach the first step of the ladder.

The inductive hypothesis says that we can get to the $k^{th}$ step of the ladder.

The inductive step says that if we can get to the $k^{th}$ step of the ladder, then we can get to step $k + 1$.

Therefore, once we get to step 1, we can get to step 2. Once we get to step 2, we can get to step 3. And so on for all steps of the ladder.

Induction for algorithms

Induction can also be used to prove the correctness of algorithms. An algorithm is simply a series of steps.

Consider the following $SORT$ algorithm which operates on a list of length $n$.

1. If $n = 1$ do nothing as the list is already sorted.
2. Find the smallest element in the list and remove it.
3. Run the $SORT$ algorithm on the remaining $n - 1$ elements.
4. Put the smallest element at the beginning of the list.

We can now prove that $SORT$ correctly sorts a list of size $n$ for $n \geq 1$.

Base case: $SORT$ does nothing on a list of size 1, which is already sorted.

Inductive hypothesis: Assume $SORT$ correctly sorts a list of size $k$.

Inductive step: We will now show that $SORT$ correctly sorts a list of size $k + 1$.

In step 3, we are left with a list of size $k$. By our inductive hypothesis, $SORT$ correctly sorts this sublist. Therefore, when we put the smallest element (which we have removed) at the beginning, our list is completely sorted as needed.

Your turn.
**Task:** You are climbing a stair case, and you are able to step either one stair at a time or two stairs at a time. Show that the number of unique ways to climb to the $n^{th}$ stair is equal to the $n + 1^{th}$ fibonacci number (let the fibonacci sequence be defined as $1, 1, 2, ...$).

Checkpoint - Call a TA over
Challenge 1: 100 dragons are sitting in a circle so that every dragon can see every other dragon.

Every dragon has green eyes, but the dragons don’t know this. No dragon knows its own eye color, and the dragons cannot talk.

Ben comes and tells the circle of dragons that at least one of them has green eyes.

If a dragon knows it has green eyes, it will turn into a human at midnight.

Prove that on night 100, all dragons will turn into humans at once.

**Hint:** Use induction! What is the base case? Ignore the number 100 and instead consider $n$ dragons and show they turn into humans on the $n^{th}$ night.

Challenge 2: We say that an infinite set $S$ is countable if there exists a bijection from $\mathbb{N}$ to $S$. Given that $\mathbb{N} \times \mathbb{N}$ is countable, prove by induction that $\mathbb{N}^k$ is countable where $k \in \mathbb{N}$.