Recitation 2
Relations, Functions, Paradox, and Infinity

Part 1: Relations

Definitions

**Defn 1:** A relation $R$ on the sets $A$ and $B$ is a subset of the Cartesian product $A \times B$.

A relation $R$ on the set $A$ is a subset of the Cartesian product $A \times A$.

Notationally, if an ordered pair $(a, b)$ is in the relation $R$, we can write $(a, b) \in R$ or $aRb$.

**Defn 2:** An equivalence relation is a relation that is reflexive, symmetric, and transitive.

**Defn 3:** A partition of a set $A$ is a collection of subsets $B_1, \ldots, B_k$ of $A$ s.t. every element of $A$ is in some subset $B_i$, but no two subsets share an element.

**Defn 4:** Let $R$ be an equivalence relation on $A$. Then the equivalence class of $a \in A$, denoted $[a]_R$, is $\{ x \mid x \in A, (x, a) \in R \}$.

**Proposition:** The equivalence classes of a relation $R$ on $A$ form a partition of $A$.

Relations Quick Guide and Common Mistakes

Discuss the following definitions and common mistakes before your proceed.

**Reflexive** A relation $R$ on set $A$ is reflexive if $(a, a) \in R$ for every $a \in A$.

Common mistake: Consider the relation $R$ on the set of students at Brown where two students are related if they took CS15 at the same time. You might think that this relation is reflexive since a student is clearly took CS15 at the same time as themself. However, there is at least one student $s$ who hasn’t taken CS15 and therefore $(s, s) \notin R$. As a result, $R$ is not reflexive. For a relation to be reflexive, $(s, s) \in R$ for every $s$ in the set.

**Symmetric** A relation $R$ on $A$ is symmetric if $\forall a, b \in A$, if $(a, b) \in R$ then $(b, a) \in R$.

Common mistake: Consider the relation $R = \{(1, 1), (2, 2)\}$ on the set $\{1, 2\}$. This relation is symmetric. Since $(1, 2) \notin R$, it is not required that $(2, 1) \in R$. 
Transitive  A relation $R$ on $A$ is *transitive* if $\forall a, b, c \in A$, if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.

Common mistake: Consider the relation $R = \{(1, 2), (1, 1)\}$ on the set $\{1, 2\}$. This relation is transitive. Can you see why?

Warm-Up

a. Consider the set $A = \{1, 2\}$.

i. Write out the Cartesian product, $A \times A$.

ii. Is $R_1 = \{(1, 1), (1, 2), (2, 2)\}$ a valid relation on $A$?

iii. Is $R_1$ reflexive? Why or why not?

iv. Is $R_1$ symmetric? Why or why not?

v. Is $R_0 = \{\}$ a valid relation on $A$?

vi. Is $R_0$ symmetric? Why or why not?

vii. Is $R_0$ transitive? Why or why not?
viii. \(R_0\) is not an equivalence relation. Why?

ix. What is the minimum number of elements in some equivalence relation on \(A\)?

**Checkpoint - Call a TA over**

b. Consider the set \(B\) of all students at Brown. For each of the following relations on \(B\), state if they are reflexive, symmetric, or transitive. If it is an equivalence relation then list the equivalence classes. **No formal proof is needed, just discuss with your group.**

i. Two students are related if they are the same age (e.g. 21).

ii. \(s_1\) and \(s_2\) are students and \((s_1, s_2) \in R\) if \(s_1\) is younger than \(s_2\).

iii. Two students are related if they are studying anthropology.

iv. Two students are related if they go to Brown.
Part 2: Functions and The Infinite

Definitions

Defn 0: A relation $R$ is a function if for every $x$ in the domain of $R$, $x$ is mapped to one and only one $y$ in the codomain of $R$.

Defn 1: The range of a function $f$ consists of all members of the codomain of $f$ that are mapped to by some member of the domain of $f$.

Defn 2: $f : X \to Y$ is an injection from set $X$ to set $Y$ if for every $y \in Y$, there is at most one $x \in X$ such that $f(x) = y$. Equivalently if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$.

An injection is often called one-to-one since you are mapping each element in $X$ to a unique element in $Y$. This guarantees that $Y$ must have at least as many elements as $X$, so $|X| \leq |Y|$.

Defn 3: $f : X \to Y$ is a surjection from set $X$ to set $Y$ if for every $y \in Y$, there is at least one $x \in X$ such that $f(x) = y$.

A surjection is often called onto since every single element in $Y$ is mapped to by $f$. This guarantees that $X$ must have at least as many elements as $Y$, so $|X| \geq |Y|$.

Defn 4: $f : X \to Y$ is a bijection if it is both an injection and surjection. Since an injection implies $|X| \leq |Y|$ and a surjection implies $|X| \geq |Y|$, a bijection guarantees $|X| = |Y|$.

Warm-Up 1

Let $A$ be the set $\{1, 2, 3\}$. Consider the following relation from $A$ to $A$, $R1: \{(1, 2), (2, 1)\}$.

a. Is $R1$ a function?

Now consider $R2$, another relation from $A$ to $A$: $\{(1, 2), (2, 1), (3, 2)\}$.

a. Is $R2$ a function?
b. If $R2$ is a function, what’s its codomain? How about its range?

Warm-Up 2

Using mapping diagrams, explain the difference among the following relations:
1) a relation $A$ on $\{0, 1\} \times \{0, 1\}$ that is not a function,
2) a relation $B \{0, 1\} \times \{0, 1\}$ that is a function but not injective, and
3) a relation $C \{0, 1\} \times \{0, 1\}$ that is both a function and injective.

Warm-Up 3

For each of the following functions, state if $f$ is an injection, surjection, or neither. Also state if it is a bijection.

Discuss your solutions.

a. $f : \{0, 1\} \rightarrow \mathbb{N}$
   $f(0) = 1, f(1) = 0$

b. $f : \mathbb{Z} \rightarrow \mathbb{Z}$
   $f(x) = x^2$

c. $f : \text{First Year Students} \rightarrow \text{First Year Dorms}$
   $f(\text{student}) = \text{dorm that student lives in}$

d. $f : \text{Brown University Students} \rightarrow \text{Countries in the World}$
   $f(\text{student}) = \text{country where student is from}$
e. \( f : \mathbb{R} \rightarrow \mathbb{R} \)
\[
f(x) = x
\]
f. Challenge \( f : \mathbb{R} \rightarrow \mathbb{R} \)
\[
f(x) = \frac{x}{2}
\]

Checkpoint - Call a TA over

To Infinity and Beyond

Introduction: Functions as Tables

It is sometimes helpful to think of a function as a table where the left column contains all elements in the domain. For example, the function \( f : \mathbb{N} \rightarrow \mathbb{N} \) where \( f(x) = x^2 \) can be represented as follows:

\[
\begin{array}{c|c}
  x & f(x) \\
  \hline
  0 & 0 \\
  1 & 1 \\
  2 & 4 \\
  3 & 9 \\
  4 & 16 \\
  \vdots & \vdots \\
\end{array}
\]

We can now redefine injectivity and surjectivity for a function \( f : X \rightarrow Y \) as follows:

- \( f \) is injective if each element in \( Y \) appears in the right column at most once.
- \( f \) is surjective if all elements of \( Y \) appear in the right column at least once.

This gives us better intuition for the important result:

**If there is a bijection from** \( X \) **to** \( Y \) **then** \( |X| = |Y| \).

If we have a unique mapping from each element in \( X \) to each element in \( Y \), and all elements of \( Y \) appear in the mapping, it must be the case that \( |X| = |Y| \).
Extending to the Infinite

The same definition applies to infinite sets. If \( A \) and \( B \) are infinite sets and there exists a bijection \( f : A \to B \) then \( A \) and \( B \) have the same cardinality.

Consider the following infinite sets:

- The natural numbers \( \mathbb{N} = \{0, 1, 2, 3, 4, \ldots\} \)
- The even natural numbers \( E = \{0, 2, 4, 6, 8, \ldots\} \)
- The odd natural numbers \( O = \{1, 3, 5, 7, 9, \ldots\} \)

Claim: \( |E| = |O| \). There are as many even numbers as odd numbers.

Proof: This is intuitive, but we can prove it by giving a bijection \( f : E \to O \).

\[
\begin{array}{c|c}
  x & f(x) = x + 1 \\
  \hline
  0 & 1 \\
  2 & 3 \\
  4 & 5 \\
  6 & 7 \\
  \vdots & \vdots \\
\end{array}
\]

However, what’s more surprising is that \( |E| = |\mathbb{N}| \).

This brings us to our first problem:

a. Show that there are just as many even natural numbers as there are natural numbers by giving a bijection \( f : \mathbb{N} \to \mathbb{E} \). You do not need to prove that this is a bijection.
**Challenge - Diagonalization**

We will now show that the reals are more numerous than the natural numbers, even if we just examine the reals between 0 and 1. Let $\mathbb{N}$ denote the natural numbers and let $\mathbb{R}_{(0,1)}$ denote the reals between 0 and 1.

Assume for the sake of contradiction that cardinality of $\mathbb{N}$ is not less than the cardinality of $\mathbb{R}_{(0,1)}$. This implies that we can construct a surjective function from $\mathbb{N}$ to $\mathbb{R}_{(0,1)}$. Let’s try to do that now!

Let’s call our mapping from $\mathbb{N}$ to $\{x \in \mathbb{R} \mid x < 0 < 1\}$, $f$. Now, consider the following table for an example surjective $f$:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.42345...</td>
</tr>
<tr>
<td>1</td>
<td>0.23456...</td>
</tr>
<tr>
<td>2</td>
<td>0.87892...</td>
</tr>
<tr>
<td>3</td>
<td>0.48897...</td>
</tr>
<tr>
<td>4</td>
<td>0.78562...</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

Imagine we’d put *every single* natural number into the lefthand side of our table, and we’d paired each natural number with some arbitrary real number between 0 and 1.

Additionally, suppose that no two entries on the left table were paired with the same entry on the right. That is, suppose that $f$ were injective.

**We’re now going to try to come up with some real number between 0 and 1 that is necessarily not on the righthand side of our table. In other words, we will show our function is not surjective.** We don’t know what each real number paired with each natural number looks like, so let’s make our notation more general.

Let $a_{i,j}$ represent the digit at the $i$th row and $j$th column of the righthand side of our table. Each $a_{i,j}$ is some arbitrary digit in $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$a_{0,0}a_{0,1}a_{0,2}a_{0,3}a_{0,4}$ ...</td>
</tr>
<tr>
<td>1</td>
<td>$a_{1,0}a_{1,1}a_{1,2}a_{1,3}a_{1,4}$ ...</td>
</tr>
<tr>
<td>2</td>
<td>$a_{2,0}a_{2,1}a_{2,2}a_{2,3}a_{2,4}$ ...</td>
</tr>
<tr>
<td>3</td>
<td>$a_{3,0}a_{3,1}a_{3,2}a_{3,3}a_{3,4}$ ...</td>
</tr>
<tr>
<td>4</td>
<td>$a_{4,0}a_{4,1}a_{4,2}a_{4,3}a_{4,4}$ ...</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
Task: Provide some a real number between 0 and 1 such that it is not equal to any $a_0, a_1, a_2, a_3, a_4, ...$. Explain why this real number necessarily does not appear on the righthand side of our table.

Hint: You’ll want to consider the following number: $.a_0, a_1, a_2, a_3, a_4, ...$. This number corresponds to diagonal through all of the entries on the righthand side of our table. How can you use this number to generate some number that is necessarily not on the righthand side of the table?

You’ve now proved that some real number between 0 and 1 is necessarily never on the righthand side of the table.

a. What does this imply about $f$?

b. What does this imply about the sizes of the natural numbers and the real numbers between 0 and 1?

c. How about the sizes of the natural numbers and the real numbers in general?

Extra Challenging Problem

We’re going to prove that if $A$ is any set, $|A| < |\mathcal{P}(A)|$.

First, we need to ensure that $|A| \leq |\mathcal{P}(A)|$. Let us define an injection $f : A \to \mathcal{P}(A)$, where $f(a) = \{a\}$. $f$ is injective (discuss why with your group).
Now, we need to show that there is no bijection between $A$ and $\mathcal{P}(A)$. For the sake of contradiction, suppose $g$ is such a bijection.

There must be some elements of $A$ which map to subsets of $A$ of which they are a member, and there must be elements of $A$ which map to subsets of $A$ of which they are not a member (for instance, whichever member of $A$ maps to the empty set). Consider the following set $S$:

$S := \{ a \in A \mid a \notin g(a) \} \subseteq A$.

$S$ must be in $\mathcal{P}(A)$, as it is a subset of $A$. Because $g$ surjective, for some $x$ in $A$, $g(x) = S$.

There are two possibilities: $x$ in $S$, and $x$ is not in $S$. Show that either case leads to a contradiction. Conclude that $|A| < |\mathcal{P}(A)|$. 