Foolproof Proof-writing

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Introduction: High School Lied to You

Who remembers writing proofs like this?

Prove the identity

\[ \cot(x) + \tan(x) = \cos(x) \csc(x) \]

\[ (\sin^2(x) \sec^2(x) + \sin^2(x) \csc^2(x)) \]

\[ \cos(x) \csc(x) \]

\[ (\sin^2(x) \cos^2(x) + \sin^2(x) \sin^2(x)) \]

\[ \cot(x) \]

Can you spot any mistakes in this proof?
Introduction: High School Lied to You

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\[
cot(x) + \tan(x) = \cos(x) \csc(x) \\
(\sin^2(x) \sec^2(x) + \sin^2(x) \csc^2(x)) = \cot(x) \\
(\sin^2(x) \cos^2(x) + \sin^2(x) \sin^2(x)) = \cot(x) \\
(\tan^2(x) + \cos^2(x) + \sin^2(x)) = \cot(x) + \tan(x)
\]
Who remembers writing proofs like this?
Prove the identity
\[ \cot(x) + \tan(x) = \cos(x) \csc(x) \left( \sin^2(x) \sec^2(x) + \sin^2(x) \csc^2(x) \right). \]
Who remembers writing proofs like this?
Prove the identity
\[ \cot(x) + \tan(x) = \cos(x) \csc(x) \left( \sin^2(x) \sec^2(x) + \sin^2(x) \csc^2(x) \right). \]

\[
\begin{align*}
\cos(x) \csc(x) \left( \sin^2(x) \sec^2(x) + \sin^2(x) \csc^2(x) \right) & \quad (1) \\
= \cot(x) \left( \frac{\sin^2(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\sin^2(x)} \right) & \quad (2) \\
= \cot(x)(\tan^2(x) + \cos^2(x) + \sin^2(x)) & \quad (3) \\
= \tan(x) + \cot(x) & \quad (4)
\end{align*}
\]
Introduction: High School Lied to You

Who remembers writing proofs like this?
Prove the identity
\[ \cot(x) + \tan(x) = \cos(x) \csc(x) \left( \sin^2(x) \sec^2(x) + \sin^2(x) \csc^2(x) \right). \]

\[ \cos(x) \csc(x) \left( \sin^2(x) \sec^2(x) + \sin^2(x) \csc^2(x) \right) \quad (1) \]
\[ = \cot(x) \left( \frac{\sin^2(x)}{\cos^2(x)} + \frac{\sin^2(x)}{\sin^2(x)} \right) \quad (2) \]
\[ = \cot(x) (\tan^2(x) + \cos^2(x) + \sin^2(x)) \quad (3) \]
\[ = \tan(x) + \cot(x) \quad (4) \]

Can you spot any mistakes in this proof?
Of course you can’t!
Introduction: High School Lied to You

Of course you can’t!

Because this isn’t a good proof.
What is a good proof?

Any ideas?
What is a good proof?

Here are some of ours:
What is a good proof?

Here are some of ours:

1. It has to be *clear*.
What is a good proof?

Here are some of ours:

1. It has to be clear.
2. It has to have good structure.
What is a good proof?

Here are some of ours:

1. It has to be *clear*.
2. It has to have good *structure*.
3. It has to *flow*.
Outline

1. Structure
2. Clarity
3. Flow
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2. Clarity
3. Flow
Structure: Proofs as Essays
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- Start with an outline.
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- Group connected ideas into paragraphs.
Structure: Proofs as Essays

▶ Start with an outline.
▶ Group connected ideas into paragraphs.
▶ Write a first draft, using complete sentences.
Structure: Proofs as Essays

- Start with an outline.
- Group connected ideas into paragraphs.
- Write a first draft, using complete sentences.
- Proofread. (Literally)
Structure: Sentence Structure

Simple sentence structure is generally easier to read.

Don't worry about sounding a little formulaic.

Use the active voice.

Example

It will be proved via contradiction...

We now prove via contradiction...
Simple sentence structure is generally easier to read.
Structure: Sentence Structure

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Example

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We now prove via contradiction...
Structure: Sentence Structure

- Simple sentence structure is generally easier to read.
- Don’t worry about sounding a little formulaic.
- Use the active voice.
- Try to only justify one thing per sentence.
Structure: Overall Structure

- Some proof types have structure that you can use to your advantage!
  - Induction
  - Element Method
  - Bijections
  - Bidirectional Proofs (If and Only If)
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Structure: Overall Structure

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Structure: Overall Structure

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- Avoid using lists inside a proof.
Some proof types have structure that you can use to your advantage!

Avoid using lists inside a proof. The description environment looks nice though!

Injectivity Proof of the injectivity of $f$ would go here. It nicely aligns the paragraphs within the proof.

Surjectivity Proof of the surjectivity of $f$ would go here.
Example Proof 1: Problem Statement

Consider the function $f : \mathbb{Z} \to \mathbb{E}$, $f(x) = 2x$. Prove that $f$ is a bijection.
Example Proof 1: Rough Draft

Proof.

It is necessary to show that $f$ is surjective and injective, or that $f(x) = f(y) \iff x = y$ $\forall x, y \in \mathbb{Z}$ and that $\forall y \in E$, $\exists x \in \mathbb{Z}$ where $f(x) = y$. For any $y \in E$ that you can think of, by definition of an even number, $y = 2x$ for some $x \in \mathbb{Z}$, since every even number can be divided by 2, no matter what. And if $f(x) = f(y)$, then $2x = 2y$ which would suggest that $x = y$. \qed
Example Proof 1: Polished

Proof.
To prove that $f$ is a bijection, we must show injectivity and surjectivity.

**Injectivity** Suppose we have $x, y \in \mathbb{Z}$ such that $f(x) = f(y)$. Then $2x = 2y$, which means $x = y$, as needed.

**Surjectivity** Consider an arbitrary $y \in \mathbb{E}$. By definition of an even number, $y = 2x$ for some $x \in \mathbb{Z}$. Thus $f(x) = 2x = y$, proving surjectivity.

Thus, $f$ is a bijection. \qed
Outline

1. Structure
2. Clarity
3. Flow
Introduction: What are you about to do?

Example

For this induction step, we will consider three cases. In order to prove that \( R \) is an equivalence relation, we need...
Clarity: Keeping the Reader Informed

▶ Introduction: What are you about to do?
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Example
For this induction step, we will consider three cases.
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Clarity: Keeping the Reader Informed

- Introduction: What are you about to do?
- Use transitions to indicate your next move.
Clarity: Keeping the Reader Informed

- Introduction: What are you about to do?
- Use transitions to indicate your next move.

**Example**

Thus, we have...
But we recall from earlier that...
Combining this with our result from case 1...
Clarity: Keeping the Reader Informed

▸ Introduction: What are you about to do?
▸ Use transitions to indicate your next move.
▸ If you use a theorem or nontrivial property to make a step, say so.
Clarity: Keeping the Reader Informed

- Introduction: What are you about to do?
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Example

...by the Fundamental Theorem of Arithmetic.
Clarity: Keeping the Reader Informed

- Introduction: What are you about to do?
- Use transitions to indicate your next move.
- If you use a theorem or nontrivial property to make a step, say so.

Example

...by the Fundamental Theorem of Arithmetic.
By definition of...
Clarity: Keeping the Reader Informed

- **Introduction:** What are you about to do?
- **Use transitions to indicate your next move.**
- **If you utilize a theorem or nontrivial property to make a step, say so.**
- **Conclusion:** What did you just do?
Clarity: Keeping the Reader Informed

- Introduction: What are you about to do?
- Use transitions to indicate your next move.
- If you utilize a theorem or nontrivial property to make a step, say so.
- Conclusion: What did you just do?

Example

...thus we have reached a contradiction.
Clarity: Keeping the Reader Informed

▶ Introduction: What are you about to do?
▶ Use transitions to indicate your next move.
▶ If you utilize a theorem or nontrivial property to make a step, say so.
▶ Conclusion: What did you just do?

Example

...thus we have reached a contradiction.
Since we have proven $P(1)$ and have shown $P(k)$ implies $P(k + 1)$,
we have shown $P(n)$ for all $n \in \mathbb{Z}^+$. 
Use notation to make your proofs simpler. Variables ($x$, $S$, $f$) are like abbreviations. Be prudent about assigning variable names.

Example:

$$S = \{ x \in P(S) \mid x \geq 3 \}$$

$$S = S \times S$$
Clarity: Notation

- Use notation to make your proof *simpler*

Variables (\(x, S, f\)) are like abbreviations. Be prudent about assigning variable names.

Example:

\[S = \{x \in P(S) \mid |x| = 3\}\]

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\[
S = \{ x \in P(S) \mid |x| = 3 \}
\]

\[
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- Be mindful about reusing variable names.
Clarity: Notation

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- Be careful about mixing symbols and words.
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- Be careful about mixing symbols and words.
  - Don’t replace a single word with a single symbol, just like you wouldn’t write “$3 + four$”.

Some symbols to keep in mind:

- $\exists$
- $\forall$
- $\therefore$
- $=$
- $\Rightarrow$
- $\Leftarrow$
Clarity: Notation

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  - Similarly, don’t write ”We know ∃ a bijection...”. Be consistent within a given context.
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  - Some symbols to keep in mind: $\exists \forall \therefore \implies =$
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Example

for all $x$ in $S$
Clarity: Notation

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    - Some symbols to keep in mind: \(\exists\ \forall\ \therefore\ \Rightarrow\ =\)

Example

- for all \(x\) in \(S\)
- \(\forall x \in S\)
Clarity: Notation

▶ Use notation to make your proof *simpler*.
▶ Variables \((x, S, f)\) are like abbreviations.
▶ Be mindful about reusing variable names.
▶ Be careful about mixing symbols and words.
  ▶ Don’t replace a single word with a single symbol, just like you wouldn’t write “3 + four”.
  ▶ Similarly, don’t write ”We know \(\exists\) a bijection...”. Be consistent within a given context.
▶ Short notation tips.
Clarity: Notation

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**Example**

prime \( p \), relation \( R \)
Clarity: Notation

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- Short notation tips.

**Example**

prime p, relation R

a₁, a₂, . . . , aₙ instead of a, b, c, . . .
Clarity: Notation

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▶ Variables \((x, S, f)\) are like abbreviations.
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Example

prime \(p\), relation \(R\)

\(a_1, a_2, \ldots, a_n\) instead of \(a, b, c, \ldots\)

\(s \in S\)
Example Proof 2: Problem Statement

Prove that there are infinitely many primes.
Example Proof 2: Rough Draft

Proof.
What if there were only finitely many primes? \( p_1, p_2, \ldots, p_n \) is the finite list of all these primes.

\[ Q = p_1 p_2 \cdots p_n + 1 \]

If \( Q \) is prime, then \( Q \) is greater than \( p_i \). \( Q \) is not \( \in \) the list of primes. \( \Rightarrow \Leftarrow \). If \( Q \) is not prime then \( p_i \mid Q \) and \( p_i \) divides \( p_1 p_2 \cdots p_n \). \( p_i \) doesn’t divide \( 1 \). \( Q - p_1 p_2 \cdots p_n = 1 \). \( \Rightarrow \Leftarrow \)
Example Proof 2: Polished

Proof.
Assume for the sake of contradiction that there are finitely many primes. Let $P = \{p_1, p_2, \ldots, p_n\}$ be the set of all primes. Now, let $q = p_1p_2\cdots p_n + 1$. We aim to show that $q$ can be neither prime nor composite. We consider the two cases:

Prime Suppose $q$ is prime. But $q > p_i$ for all $i$, meaning that $q \not\in P$. This contradicts our definition of $P$.

Composite Suppose $q$ is not prime; by the Fundamental Theorem of Arithmetic, $q$ can be factored into primes. Consider $p_i$, one of these prime factors. Since $p_i \mid q$ and $p_i \mid p_1p_2\cdots p_n$, we know that $p_i \mid (q - p_1p_2\cdots p_n)$. But $q - p_1p_2\cdots p_n = 1$, meaning that $p_i \mid 1$. This is a contradiction.

Thus, we have proven that there cannot be finitely many primes. \qed
Outline

1. Structure
2. Clarity
3. Flow
You do not need to restate definitions.

Example: We are given that $B_1, \ldots, B_k$ is a partition of $U$ into distinct blocks such that every element in $U$ is in some block.
Flow: Avoiding Redundancy

- You do not need to *restate* definitions.
Flow: Avoiding Redundancy

- You do not need to *restate* definitions.

**Example**

We are given that \( B_1, \ldots, B_k \) is a partition of \( U \) into distinct blocks such that every element in \( U \) is in some block.
Flow: Avoiding Redundancy

- You do not need to *restate* definitions.
- Exception: Recalling an earlier proven point or citing a sub-result out of context.
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Example

…it is a bijection. Because it is surjective…
Flow: Avoiding Redundancy

- You do not need to *restate* definitions.
- Exception: Recalling an earlier proven point or citing a sub-result out of context.

Example

...it is a bijection. Because it is surjective...
Recall that $R$ is an equivalence relation. By the transitivity of $R$...
Flow: Avoiding Redundancy

- You do not need to *restate* definitions.
- Exception: Recalling an earlier proven point or citing a sub-result out of context.
- Examples are rarely very useful.
Flow: Avoiding Redundancy

- You do not need to *restate* definitions.
- Exception: Recalling an earlier proven point or citing a sub-result out of context.
- Examples are rarely very useful.
Flow: Using Meaningful Transitions

Example
Consider two distinct elements \( a_1, a_2 \in A \). Without loss of generality, \( a_1 < a_2 \).
Flow: Using Meaningful Transitions

- Hence, thus, therefore.

Example: Consider two distinct elements $a_1, a_2 \in A$. Without loss of generality, $a_1 < a_2$. 
Flow: Using Meaningful Transitions

- Hence, thus, therefore.
- We need to show...
  In order to prove...
Flow: Using Meaningful Transitions

- Hence, thus, therefore.
- We need to show...
  In order to prove...
- It suffices to show...
Flow: Using Meaningful Transitions

- Hence, thus, therefore.
- We need to show...
  In order to prove...
- It suffices to show...
- ...as needed.
Flow: Using Meaningful Transitions

- Hence, thus, therefore.
- We need to show...
  In order to prove...
- It suffices to show...
- ...as needed.
- Suppose...
Flow: Using Meaningful Transitions

- Hence, thus, therefore.
- We need to show...
  In order to prove...
- It suffices to show...
- ...as needed.
- Suppose...
- Let $x$...
Flow: Using Meaningful Transitions

- Hence, thus, therefore.
- We need to show...
  In order to prove...
- It suffices to show...
- ...as needed.
- Suppose...
- Let $x$...
- Consider...
Flow: Using Meaningful Transitions

- Hence, thus, therefore.
- We need to show...
  In order to prove...
- It suffices to show...
- ...as needed.
- Suppose...
- Let $x$...
- Consider...
- Recall...
Flow: Using Meaningful Transitions

- Hence, thus, therefore.
- We need to show...
  In order to prove...
- It suffices to show...
- ...as needed.
- Suppose...
- Let \( x \)...
- Consider...
- Recall...
- In particular...
Flow: Using Meaningful Transitions

- Hence, thus, therefore.
- We need to show...
  In order to prove...
- It suffices to show...
- ...as needed.
- Suppose...
- Let $x$...
- Consider...
- Recall...
- In particular...
- Without loss of generality (wlog)
Flow: Using Meaningful Transitions

- Hence, thus, therefore.
- We need to show...
  In order to prove...
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Example

Consider two distinct elements $a_1, a_2 \in A$. Without loss of generality, $a_1 < a_2$. 
Flow: Using Meaningful Transitions

- Hence, thus, therefore.
- We need to show...
  In order to prove...
- It suffices to show...
- ...as needed.
- Suppose...
- Let $x$...
- Consider...
- Recall...
- In particular...
- Without loss of generality (wlog)
- Clearly, obviously, trivially
Example Proof 3: Problem Statement

Consider the following relation on the set of integers:
\[ \forall a, b \in \mathbb{Z}, (a, b) \in R \text{ if and only if } a \text{ and } b \text{ have the same remainder when divided by 3.} \]
Prove that \( R \) is transitive.
Example Proof 3: Rough Draft

Proof.
We know that dividing integers by integers will yield integer remainders, by properties of division. So let $r_a$ be the remainder when you divide $a$ by 3. Similarly for $r_b$ and $r_c$ with $b, c$.

Definition of transitivity:

\[(a, b), (b, c) \in R \implies (a, c) \in R \quad \forall a, b, c \in \mathbb{Z}\]

so we need this to be true to show transitivity. (e.g. $(1, 2), (2, 3) \in R \implies (1, 3) \in R$.)

Notice $(a, b) \in R \implies r_a = r_b$ and $(b, c) \subseteq R \implies r_b = r_c$ so $r_a = r_c$.

So $R$ is transitive because $(a, c) \in R$ for all $(a, b), (bc) \in R$. \qed
Example Proof 3: Polished

Proof.
For transitivity to hold, we need

\[(a, b), (b, c) \in R \implies (a, c) \in R \quad \forall a, b, c \in \mathbb{Z}.\]

Let \(r_a\), \(r_b\), and \(r_c\) be the remainders when you divide \(a\), \(b\), and \(c\) by 3, respectively. Since \((a, b) \in R\), we know that \(r_a = r_b\). Since \((b, c) \in R\), we know that \(r_b = r_c\). Thus, we have \(r_a = r_c\). By definition of the relation \(R\), \((a, c) \in R\), as needed.

Thus, we have shown that \(R\) is transitive.
Outline

1. Structure
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3. Flow