Practice Problems 2

Problem 1

Let $p$ be a prime number. An integer $k$ is a self-inverse modulo $p$ iff $k^2 \equiv 1 \mod p$. Find all integers that are self-inverses modulo $p$.

Problem 2

How many non-negative integers must you pick in order to be sure that at least two of them have the same remainder when divided by 15?

Problem 3

How many positive integers less than or equal to 210 are not divisible by any of 2, 3, or 7? (In other words, how many numbers do not have 2, 3, or 7 as prime factor?

Problem 4

How many permutations of the digits $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ contain the sequence 024 or the sequence 456? (Note: “or” in this case is NOT an exclusive-or.)

Problem 5

Add parentheses to the following expressions to make them true (1 represents true, and 0 represents false).

- a. $0 \lor 1 \lor 1 \implies 1 \land 1 \land 1 \lor 0$
- b. $0 \lor 0 \land 1 \land 0 \land 1 \land 1 \lor 1$
- c. $0 \land 1 \lor 1 \iff 0 \implies 0$
- d. $0 \lor 1 \implies 0 \land 0 \implies 1$

Problem 6

Bacan, Lettuce, Tomato and Mayo are riding the subway. They are board and decide to play a game using some sandwiches. The game has the following rules:
I. Lettuce loses if and only if Mayo eats the sandwich.
II. If Mayo stands up, then Lettuce throws sandwiches at Tomato.
III. If Lettuce throws sandwiches at Tomato, then Bacan yells at Lettuce.
IV. Mayo stands up if and only if they do not eat the sandwich.
V. If Bacan yells at Lettuce, then Mayo does not eat the sandwich.
VI. Every player must either win or lose.
VII. If Lettuce does not throw sandwiches at Tomato, then Tomato wins.
VIII. Only one player can win.

Prove the following statements:
a. Lettuce wins if and only if they throws sandwiches at Tomato.
b. Mayo eats the sandwich if and only if Bacan does not yell at Lettuce.
c. The statement ”Mayo cannot win” is a tautology.

Problem 7

Design a combinatorial circuit (using only AND, OR and NOT gates) that checks if two 0/1 strings of length 4 are equal. Hence the circuit should have 8 inputs and should output 1 if the strings are the same and 0 if they are not the same.

Problem 8

Give a compound proposition (using the operations AND, OR, NOT) built from three propositions $p$, $q$, and $r$ such that the compound proposition is True only if: $p$ and $q$ are both True and $r$ is False.

Problem 9

Consider the following way to express a non-negative integer $m$:

$$m = a_k3^k + a_{k-1}3^{k-1} + \ldots + a_13 + a_0$$

where $a_i \in \{0, 1, 2\}$ for all $i$.

(a) Express the numbers 7, 8 and 9 in the form above.
(b) Prove that every non-negative integer $m$ may be expressed in this form.
(c) Prove that $m$ is even iff the sum of the $a_i$s is even. (Hint: work $\text{mod} \ 2$)
Problem 10

Consider 7 digit integers (which do not contain a 0 in the 1st digit). For each of the following conditions, count the number of such integers, and prove your answer.

(a) The number has all even digits.
(b) Either the number has no repeated digits, or all digits are the same.
(c) No two consecutive digits are the same.
(d) The digits are in strictly increasing order.

(If the digits are $D_1D_2D_3D_4D_5D_6D_7$, then $D_1 < D_2 < \ldots < D_7$.)